Quiz 10
Sample Solutions

Name: ______________________________

Student Number: ______________________________

In the space provided, please write your solutions to the following exercises. *Fully explain your reasoning.* Remember to use good notation and full sentences.

Good Luck!

1. Let R be a ring.

   (i) Complete the following definition: R is an *integral domain* if

   Solution: R is a commutative ring with identity such that for every $a, b \in R$ such that $ab = 0$, either $a = 0$ or $b = 0$.

   (ii) An element x in R is called an *idempotent* if $x^2 = x$. Prove that the only idempotents in an integral domain are 0 and 1.

   Solution: Suppose that R is an integral domain. Let $x \in R$ be an idempotent. Then $x^2 = x$ and so

   $$0 = x^2 - x = x(x-1).$$

   Since R is an integral domain, we must have that either $x = 0$ or $x - 1 = 0$. Hence, $x = 0$ or $x = 1$.

 Turn over for Exercises (2) and (3) ...

2. Let R be a ring. Define the *center* of R to be

$$Z(R) = \{a \in R \mid ar = ra \text{ for all } r \in R\}.$$

Prove that $Z(R)$ is a subring of R.

Solution: First note that $0 \in Z(R)$ since

$$0r = r0 = 0$$

for all $r \in R$. Thus, $Z(R) \neq \emptyset$. Also, if $x, y \in Z(R)$ then for all $r \in R$ we have

- $(xy)r = x(yr) = x(ry) = (xr)y = (rx)y = r(xy)$
- $(x - y)r = xr - yr = rx - ry = r(x - y)$

and so xy and $x - y$ are both elements of $Z(R)$. Therefore, by the Subring Test, $Z(R)$ is a subring of R.

3. Determine (with justification!) the characteristic of the field formed by the set of matrices

$$F = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

with entries in $\mathbb{Z}_2$.

Solution: Note that the identity of F is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, the order of the identity element (in the additive abelian group $(F, +)$) is 2. By Lemma 16.18 of our textbook, this implies that the characteristic of F is 2.