MATH 2020: Algebra 1
Tutorial 7 Worksheet – March 5, 2018

Question 1. Let

$$H = \left\{ A \in GL_2(\mathbb{R}) \mid A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right\},$$

which is a subgroup of $GL_2(\mathbb{R})$. Prove that $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ is isomorphic to H.

Question 2. Prove that $\mathbb{Q}$ is not isomorphic to $\mathbb{Z}$.

Question 3. Let G be a group of order 20. If G has subgroups H and K of orders 4 and 5 respectively such that $hk = kh$ for all $h \in H$ and $k \in K$, prove that G is the internal direct product of H and K.

Question 4. Let $\phi : G_1 \to G_2$ and $\psi : G_2 \to G_3$ be isomorphisms. Show that ϕ^{-1} and $\psi \circ \phi$ are both isomorphisms. Using these results, show that the isomorphism of groups determines an equivalence relation on the class of all groups.

Question 5. An automorphism of a group is an isomorphism with itself. The set of all automorphisms of a group is denoted $\mathbf{Aut(G)}$. Prove that $\mathrm{Aut}(G)$ is a subgroup of S_G, the group of permutations of G.

Question 6. Let G be a group and $g \in G$. Define a map $i_g : G \to G$ by $i_g(x) = gxg^{-1}$. Prove that i_g defines an automorphism of G. Such automorphisms are called inner automorphisms. The set of all inner automorphisms of a group G is denoted $\mathbf{Inn(G)}$

Question 7. Prove that $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$.

Question 8. Let H_1 and H_2 be subgroups of G_1 and G_2 respectively. Prove that $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$.