MATH 2020: Algebra 1
Tutorial 5 Worksheet – February 5, 2018

Question 1. Suppose that G is a group and let $a, b \in G$, with $|a| = m$, $|b| = n$ and $\gcd(m, n) = 1$.
(a): Show that $\langle a \rangle \cap \langle b \rangle$ is a subgroup of both $\langle a \rangle$ and $\langle b \rangle$.
(b): Use the Fundamental Theorem of Cyclic Groups to conclude that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

Note that this is the same as question 7 from last week's tutorial. The idea is to see a different approach to solving it.

Question 2. Let p and q be distinct primes. How many generators does $\mathbb{Z}_{pq}$ have?

Question 3. Let a be an element in a group G. What is a generator for the subgroup $\langle a^m \rangle \cap \langle a^n \rangle$?

Question 4. Denote a cycle by $(a_1 a_2 ... a_n)$. Find its inverse.

Question 5. Let $\sigma = \sigma_1 \sigma_2 ... \sigma_m \in S_n$ be a product of disjoint cycles.
(a): Prove that the order of cycle σ_i is its length.
(b): Denote $\ell := lcm(\text{length}(\sigma_1), \text{length}(\sigma_2), ..., \text{length}(\sigma_m))$. Show that $\sigma^\ell = id$.
(c): Show that if $\sigma^k = id$, then $\ell | k$. Conclude that $|\sigma| = \ell$.

Question 6. Let $\sigma \in S_n$ be a cycle. Prove that σ can be written as the product of at most $n - 1$ transpositions.

Question 7. Let $\sigma \in S_n$. If σ is not a cycle, prove that σ can be written as a product of at most $n - 2$ transpositions.

Question 8. Let $\alpha \in S_n$ for $n \geq 3$. If $\alpha\beta = \beta\alpha$ for all $\beta \in S_n$, prove that α must be the identity permutation. This shows that the center of S_n is the trivial subgroup, i.e. $Z(S_n) = \{id\}$, for all $n \geq 3$.

Question 9. Compute each of the following:
(a): $(12)(1253)$
(b): $(143)(23)(24)$
(c): $[(12)(34)(12)(47)]^{-1}$
(d): $(12357)^{-1}$