# MATH 2020: Algebra 1
# Tutorial 3 Worksheet – January 22, 2018

**Question 1.** Define the *least common multiple* of two nonzero integers $a$ and $b$, denoted by $lcm(a, b)$, to be the nonnegative integer $m$ such that

(1) $a|m$ and $b|m$

(2) If for some integer $n$, we have both $a|n$ and $b|n$, then $m|n$.

Prove that $gcd(a, b) = 1$ if and only if $lcm(a, b) = ab$.

**Question 2.** Let $n \in \mathbb{N}$. Use the division algorithm to prove that every integer is congruent mod $n$ to precisely one of the integers $0, 1, ..., n-1$. Conclude that if $r$ is an integer, then there is exactly one $s \in \mathbb{Z}$ such that $0 \leq s < n$ and $[r] = [s]$. Hence, the integers are indeed partitioned by congruence mod $n$.

**Question 3.** Prove that $gcd(a, c) = gcd(b, c) = 1$ if and only if $gcd(ab, c) = 1$ for integers $a, b$, and $c$.

**Question 4.** Let $S := \mathbb{R}\backslash\{-1\}$ and define an operation on $S$ by $a * b = a + b + ab$.

**(a)** Prove that $*$ is a binary operation.

**(b)** Prove that $(S, *)$ is an abelian group.

**Question 5.** Prove or disprove that every group with 6 elements is abelian.

**Question 6.** Let $G$ be a group and $g \in G$. Prove that if $g^2 = g$, then $g = e$, where $e$ is the identity element.

**Question 7.** Let $G$ be a finite group with an even number of elements. Prove that there exists $a \in G$ such that $a \neq e$ and $a^2 = e$. *Hint:* The statement of question 6 is useful.

**Question 8.** Prove that a group $G$ is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

**Question 9.** Given the groups $\mathbb{R}^*$ and $\mathbb{Z}$ where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$, let $G = \mathbb{R}^* \times \mathbb{Z}$ and define a binary operation $\circ$ on $G$ by $(a, m) \circ (b, n) = (ab, m + n)$. Show that $G$ is a group under this operation.