MATH 2020: Algebra 1
Tutorial 2 Worksheet – January 15, 2018

Question 1. Let S and T be finite sets of the same cardinality and $f : S \to T$. Prove that f is one-to-one if and only if f is onto.

Question 2. Define $S = \{1, 2, 3\}$. How many unique functions f are there such that $f : S \to S$? How many of them are bijections?

Question 3. Let $f : A \to B$ and $g : B \to C$ be maps.
 (a) If f and g are both one-to-one, show that $g \circ f$ is one-to-one.
 (b) If $g \circ f$ is one-to-one and f is onto, show that g is one-to-one.
 (c) If $g \circ f$ is onto and g is one-to-one, show that f is onto.

Question 4. Prove the relation defined on $\mathbb{R}^2$ by $(x_1, y_1) \sim (x_2, y_2)$ if $x_1^2 + y_1^2 = x_2^2 + y_2^2$ is an equivalence relation. What are the equivalence classes?

Question 5. Define a relation $\sim$ on $\mathbb{R}$ given by $x \sim y$ if $x - y \in \mathbb{Q}$. Prove or disprove: $\sim$ is an equivalence relation.

Question 6. Define a relation $\sim$ on $\mathbb{R}^2$ by stating that $(a, b) \sim (c, d)$ if and only if $a^2 + b^2 \leq c^2 + d^2$. Show that $\sim$ is reflexive and transitive but not symmetric.

Question 7. For $a_i \in \mathbb{R}$ with $a_i > 0$ for all i, show that for $n \in \mathbb{N}$,

$$\sqrt[n]{a_1 a_2 ... a_n} \leq \frac{1}{n} \sum_{k=1}^{n} a_k.$$

Question 8. Let X be a set. Define the *power set* of X, denoted by $\mathcal{P}(X)$, to be the set of all subsets of X. For example,

$$\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

For every positive integer n, show that a set with exactly n elements has a power set with exactly 2^n elements.

Question 9. For $n \in \mathbb{N}$, show that $5 \mid (6^n - 1)$.