MATH 2020: Algebra 1
Tutorial 10 Worksheet – March 26, 2018

Question 1. Let R be a ring of 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix},$$

where $a, b \in \mathbb{R}$.

(a): Show that R has no identity. You may assume without proof that R is a ring.
(b): Show that there is a subring S of R that has an identity.

Question 2. Prove that the Gaussian integers, $\mathbb{Z}[i]$, are an integral domain.

Question 3. Let R be a ring. Define the *center* of R to be

$$Z(R) = \{a \in R \mid ar = ra \text{ for all } r \in R\}.$$

Prove that $Z(R)$ is a subring of R.

Question 4. Determine (with justification) the characteristic of the field formed by the set of matrices

$$F = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

with entries in $\mathbb{Z}_2$.

Question 5. Let R be a ring with a collection of subrings $\{R_\alpha\}$. Prove that $\cap R_\alpha$ is a subring of R. Give an example to show that the union of two subrings is not necessarily a subring.

Question 6. Let S be a nonempty subset of a ring R. Prove that there is a subring R' of R that contains S.

Question 7. Let R and S be arbitrary rings. Show that their Cartesian product is a ring if we dfine addition and multiplication in $R \times S$ by

$$(r, s) + (r', s') = (r + r', s + s'), \quad \text{and} \quad (r, s)(r', s') = (rr', ss').$$

Question 8. An element $x \in R$ is called an *idempotent* if $x^2 = x$. Prove that the only idempotents in an integral domain are 0 and 1.