Problem Set 5

Due: 9:00 a.m. on Wednesday, October 9

Instructions: MATH 7470 students should submit solutions to all of the following problems and MATH 4470 students should submit solutions to only those marked with a "U". A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. You may submit your solutions either in-class or to the Department of Mathematics (*with date and time of submission noted*).

Exercises: For this Problem Set, let R be a commutative ring with identity.

1U. (The Five Lemma) Consider the following commutative diagram of R-module homomorphisms with exact rows:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

(a) Prove that if α_1 is surjective and α_2, α_4 are injective, then α_3 is injective.

(b) Prove that if α_5 is injective and α_2, α_4 are surjective, then α_3 is surjective.

2. Consider the following commutative diagram of R-module homomorphisms with exact rows:

$$\begin{array}{cccccccc} & & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & & \end{array}$$

(a) Prove that there is an exact sequence $\mathrm{Ker}(\alpha_1) \to \mathrm{Ker}(\alpha_2) \to \mathrm{Ker}(\alpha_3)$.

(b) Prove that there is an exact sequence $B_1/\mathrm{Im}(\alpha_1) \to B_2/\mathrm{Im}(\alpha_2) \to B_3/\mathrm{Im}(\alpha_3)$.