Problem Set 4

Due: 9:00 a.m. on Wednesday, October 2

Instructions: MATH 7470 students should submit solutions to all of the following problems and MATH 4470 students should submit solutions to only those marked with a "U". A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. You may submit your solutions either in-class or to the Department of Mathematics (*with date and time of submission noted*).

Exercises: From the textbook *Abstract Algebra*, 3rd edition, by David S. Dummit and Richard M. Foote.

Unless otherwise instructed, throughout assume that R is a ring with identity $1 \neq 0$ and that M is a left R-module.

1U. Let R be a ring with identity and let M be a unital right R-module.

- (i) Prove that every element of $M \otimes_R R$ is of the form $m \otimes_R 1$.
- (ii) Prove that there is an Abelian group isomorphism $F : M \otimes_R R \to M$ such that $F(m \otimes_R r) = mr$ for all $m \in M$ and $r \in R$.

2U. (Dummit and Foote §10.4 #16) Suppose that R is commutative and let I and J be ideals of R, so R/I and R/J are naturally R-modules.

- (a) Prove that every element of $R/I \otimes R/J$ can be written as a simple tensor of the form $(1 \mod I) \otimes (r \mod J)$.
- (b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r \mod I) \otimes (r' \mod J)$ to $rr' \mod (I + J)$.

3. (Dummit and Foote §10.4 #20) Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. Show that the element $2 \otimes 2 + x \otimes x$ in $I \otimes_R I$ is not a simple tensor, i.e., cannot be written as $a \otimes b$ for some $a, b \in I$.