Problem Set 11
Due: 9:00 a.m. on Wednesday, November 27

Instructions: MATH 7470 students should submit solutions to all of the following problems and MATH 4470 students should submit solutions to only those marked with a "U". A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. You may submit your solutions either in-class or to the Department of Mathematics (*with date and time of submission noted*).

Exercises: For this Problem Set, assume that all rings are non-zero, commutative, and contain an identity $\neq 0$.

1. Let B be a commutative ring and A a subring of B (so that $1 \in A$). An element x of B is said to be *integral over* A if x satisfies an equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

where the a_i are elements of A. You may assume that the set C of elements of B which are integral over A is a subring of B containing A. We say that B is *integral* over A if $C = B$ and we say that A is *integrally closed* in B if $C = A$. Suppose that B is integral over A.

(iU) Let $\mathfrak{b}$ be an ideal of B and $\mathfrak{a} = A \cap \mathfrak{b}$. Prove that $B/\mathfrak{b}$ is integral over $A/\mathfrak{a}$.

(iiU) Prove that the field of fractions of B is integral over the field of fractions of A.

(iii) Suppose further that $A \subseteq B$ are integral domains. Prove that B is a field if and only if A is a field.

(iv) Let $\mathfrak{q}$ be a prime ideal of B and let $\mathfrak{p} = \mathfrak{q} \cap A$. Prove that $\mathfrak{q}$ is maximal if and only if $\mathfrak{p}$ is maximal.