Problem Set 6

Due: 10:00 a.m. on Thursday, October 24

Instructions: Submit solutions to all of the following exercises. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. You may submit your solutions either in-class or to the Department of Mathematics (*with date and time of submission noted*).

Exercises: Be sure to show all of your work and fully justify your answers and reasoning.

1. Consider the digraph $G(V, E)$ (with loops) given by

$$V = \{a, b, c\} \quad \text{and} \quad E = \{(a, a), (a, b), (b, b), (b, c), (c, c)\}.$$

   (a) Sketch a representation for G.

   (b) How many $a \to b$ paths are there of length 1? 2? 3? 4? Justify your answer by listing the paths.

   (c) How many $a \to c$ paths are there of length 1? 2? 3? 4? Justify your answer by listing the paths.

   (d) Let p_n denote the number of paths $a \to b$ of length n and let q_n denote the number of $a \to c$ paths of length n for $n \geq 1$. Explain why $q_n = q_{n-1} + p_{n-1}$.

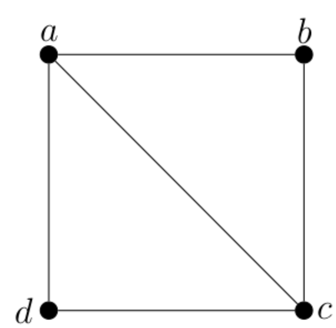
   (e) It turns out that $p_n = n$ for all $n \geq 1$ (you may accept this without proof). Guess and prove (by induction) a formula for q_n.

2. Suppose $V \subseteq \mathbb{Z}^+$. Let us define the *divisibility* graph of V to be the digraph such that for $a, b \in V$, (a, b) is an arc if and only if a divides b and $a \neq b$.

   (a) Draw a representation for the divisibility graph D for $V = \{2, 3, \ldots, 12\}$.

   (b) Let G be the underlying graph of D from part (a). How many connected components does G have? Explicitly show what the components are.

3. Let G be the following graph:



   (a) Find the adjacency matrix A of G.

   (b) Compute A^2, A^4 and A^6.

   (c) How many paths $a \to b$ of length 6 are there in G?

(d) How many cycles $c \to c$ of length 6 are there in G?

(e) How many cycles of length 6 are there in G? Here, treat the same cycle with different starting points as different: for example, $a{-}{-}b{-}{-}c{-}{-}a$ is different from $b{-}{-}c{-}{-}a{-}{-}b$. Justify your answer.

4. Let A be the adjacency matrix of a graph G and

$$A^3 = \begin{bmatrix} 2 & 5 & 5 & 2 & 0 & 0 \\ 5 & 4 & 5 & 5 & 0 & 0 \\ 5 & 5 & 4 & 5 & 0 & 0 \\ 2 & 5 & 5 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) How many paths of length 3 are there from vertex 3 to vertex 1?

(b) How many connected components does the graph have? Completely justify your reasoning.

5. Let A be the adjacency matrix of a simple graph G of order $n \geq 1$. Prove that G is connected if and only if $A + A^2 + \cdots + A^n$ has no zero entries.

6. An acyclic graph is sometimes called a *forest*. Let G be a forest of order $n \geq 1$ with K connected components. Determine a formula for the size of G. Fully justify your reasoning.