Problem Set 3

Due: 10:00 a.m. on Thursday, September 26

Instructions: Submit solutions to all of the following exercises. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets.* You may submit your solutions either in-class or to the Department of Mathematics (*with date and time of submission noted*).

Exercises: Be sure to show all of your work and fully justify your answers and reasoning.

1. Consider the following statements:

   A. All integers are real numbers.
   B. No irrational number is an integer.
   C. Some real numbers are not integers.

   (a) Which of these statements involve an existential quantifier?
   (b) Which of these statements involve a universal quantifier?
   (c) Express the negation of the quantified statement "Some of Susan's jokes are funny."

2. The *Fibonacci numbers* is a sequence $\{f_n\}_{n\geq 1}$ defined as follows:

$$f_n = \begin{cases} 1 & \text{for } n = 1 \text{ or } 2 \\ f_{n-1} + f_{n-2} & \text{for } n \geq 3 \end{cases}$$

   (a) Use strong induction to prove the formula

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

   (b) Unbelievably at first, this awful-looking formula yields an integer value for every positive integer n. Explain why this is true.

3. A polynomial in one variable with real coefficients is said to be *irreducible* if it cannot be expressed as the product of two polynomials of smaller degree with real coefficients. For example, x^2+1 is irreducible, but $x^2-1 = (x+1)(x-1)$ is not. Use complete induction to prove that any non-zero real polynomial can be factored as a product of irreducible polynomials or is itself irreducible. (*Hint:* Look at the proof of the Fundamental Theorem of Arithmetic)

   Note: There are two things of note here which don't follow from the proof.

   - Like with the Fundamental Theorem of Arithmetic, the factoring is unique as long as you ignore factoring out of constants.
   - In the real numbers, the only irreducible polynomials are constant, linear and quadratic polynomials using that any complex roots of a polynomial with real coefficients come in pairs. This however is not true with integers since $x^3 + 2$ cannot be factored into two polynomials with integer coefficients.