Problem Set 11

Due: 10:00 a.m. on Thursday, December 5

Instructions: Submit solutions to all of the following exercises. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. You may submit your solutions either in-class or to the Department of Mathematics (*with date and time of submission noted*).

Exercises: Be sure to show all of your work and fully justify your answers and reasoning.

1. Consider a cyclic group of order 5, say $C_5 = (\{e, g, g^2, g^3, g^4\}, \cdot)$ where $g^5 = e$. Let $\alpha : C_5 \to C_5$ be an isomorphism such that $\alpha(g) = g^2$. Determine the image of each element of C_5 under α.
2. Let G_1 be the group of real numbers under the binary operation of addition and G_2 be the group of positive real numbers under the binary operation of multiplication. You may assume that G_1 and G_2 are indeed groups without verification. Define the function $\varphi : G_1 \to G_2$ by $\varphi(x) = 2^x$. Show that φ is a group isomorphism.
3. The **group of quaternions** Q is defined to be the set $\{\pm 1, \pm i, \pm j, \pm k\}$ where $(-1)^2 = 1$, -1 commutes with all elements, $-x$ ix shorthand for $(-1)x, i^2 = j^2 = k^2 = -1$ and $ij = k, jk = i, ki = j$.
   (a) Find ji, ik and kj.
   (b) Is Q isomorphic to D_8, C_8 or neither? Explain your answer. (*Hint:* Isomorphic groups have the same number of elements of any given order.)
4. Consider the group $G = (\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}, \oplus)$ where $a \oplus b = a + b \mod 10$.
   (a) Find the orders of the elements 2 and 7.
   (b) Prove that $H = \{0, 2, 4, 6, 8\}$ is a subgroup of G.
5. Prove that if $(ab)^2 = a^2b^2$ in a group G, then $ab = ba$.
6. Let $C_8 = (\{e, g, g^2, g^3, g^4, g^5, g^6, g^7\}, \cdot)$ be a cyclic group of order 8. $H = \{e, g^4\}$ is a subgroup of G. Find all the left cosets of H in G.
7. Let $C_n = (\{e, g, g^2, \ldots, g^{n-1}\}, \cdot)$ with $g^n = e$ be a cyclic group of order $n \geq 2$ where n is a prime number. Use Lagrange's Theorem to show that the only two subgroups of C_n are $\{e\}$ and C_n.