Problem Set 10

Due: 10:00 a.m. on Thursday, November 28

Instructions: Submit solutions to all of the following exercises. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. You may submit your solutions either in-class or to the Department of Mathematics (*with date and time of submission noted*).

Exercises: Be sure to show all of your work and fully justify your answers and reasoning.

1. Let U be a set and let S be the set of subsets of U. Recall that the *symmetric difference* of $A, B \in S$ is defined as
$$A\Delta B = (A \cup B) \setminus (A \cap B).$$
   (a) Prove that $G = (S, \Delta)$ is a group.
   (b) Prove that $G = (S, \Delta)$ is an Abelian group.

2. Consider the binary operation 8 defined as $a * b = ab \bmod 15$.
   (a) Find the set $S \subseteq \{0, 1, \dots, 14\}$ such that $G = (S, *)$ is a group. (It is not necessary to prove that G is a group. *Hint:* Find the invertible elements.)
   (b) Write out the multiplication table for G.
   (c) Is the group G Abelian? Justify your answer.
   (d) Is the group G cyclic? Justify your answer.

3. Let $GL(2, \mathbb{R})$ denote the set of all 2×2 matrices with real entries and non-zero determinant. You may assume the fact that $GL(2, \mathbb{R})$ is an infinite, non-Abelian group under the binary operation of matrix multiplication. In addition, you may assume the fact that $\mathbb{R}\backslash\{0\}$ is an infinite group under the binary operation of multiplication. Define the function $\varphi : GL(2, \mathbb{R}) \to \mathbb{R}\backslash\{0\}$ by $\varphi(A) = \det(A)$. Show that φ is a group homomorphism.

4. Let G_1 be the group of real numbers under the binary operation of addition and G_2 be the group of positive real numbers under the binary operation of multiplication. You may assume that G_1 and G_2 are indeed groups without verification. Define the function $\varphi : G_1 \to G_2$ by $\varphi(x) = 2^x$. Show that φ is a group homomorphism.