Vector Spaces

Motivating Examples: Central to this course is an in-depth study of vector spaces. Before we state the formal definition (which may seem quite abstract at first), we look at some motivating examples.

1. Let
$$\mathbb{R}^n = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} : a_1, \ldots, a_n \in \mathbb{R} \right\}.$$
   We define "vector addition" and "scalar multiplication" as

   You can visualize $\mathbb{R}^2$ as the usual Cartesian plane and $\mathbb{R}^3$ as a 3-dimensional space.

2. Let $M_{m\times n}(\mathbb{R})$ denote the set of all $m \times n$ matrices with real entries. That is,

   Here we define "vector addition" and "scalar multiplication" entry-wise where scalars are from the real numbers $\mathbb{R}$. For example,

3. Let $\mathcal{P}_n(\mathbb{R})$ denote the set of polynomials of degree at most n with real coefficients. That is,

$$\mathcal{P}_n(\mathbb{R}) = \{p(x) = a_0 + a_1x + \cdots + a_nx^n \mid a_i \in \mathbb{R}\}.$$

As with the previous two examples, polynomials are objects that you can add and multiply by scalars. Precisely, we define

$$(a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

Also, if $\alpha \in \mathbb{R}$ is a scalar then

$$\alpha(a_0 + a_1x + \cdots + a_nx^n) = \alpha a_0 + \alpha a_1x + \cdots + \alpha a_nx^n.$$

Note: $\mathbb{R}^n, M_{m\times n}(\mathbb{R})$ and $\mathcal{P}_n(\mathbb{R})$ have many similarities! For example,

- we can add two elements in each one and get an element in the same space (this is called *closure under addition*)
- we can scalar multiply an element and get an element in the same space (this is called *closure under scalar multiplication*)

Notation: Throughout the course, we will let $\mathbb{F}$ denote either:

- the real numbers $\mathbb{R}$
- the complex numbers $\mathbb{C}$

It is now time to formally define vector spaces!

Definitions: Let V be a set and fix the field $\mathbb{F}$.

1. An **addition operation** on V is a function that assigns an element

2. A **scalar multiplication** operation on V is a function that assigns an element

Definition: A **vector space** over a field $\mathbb{F}$ is a set V equipped with addition and scalar multiplication operations such that for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$ and scalars $r, s \in \mathbb{F}$ the following properties hold:

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
3. There is an unique element $\mathbf{0}$ in V such that
$$\mathbf{0} + \mathbf{a} = \mathbf{a} + \mathbf{0} = \mathbf{a}.$$
4. For every element $\mathbf{a} \in V$ there exists a unique element, denoted $-\mathbf{a}$, in V such that
$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}.$$
5. $1\mathbf{a} = \mathbf{a}$
6. $(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$
7. $r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$
8. $(rs)\mathbf{a} = r(s\mathbf{a})$

The elements of V are called

Note: $\mathbb{R}^n, M_{m\times n}(\mathbb{R})$ and $\mathcal{P}_n(\mathbb{R})$ are all vector spaces over the field $\mathbb{R}$.

Example: Let's check some of the vector space axioms for $\mathbb{R}^3$.

Proposition: Let V be a vector space over a field $\mathbb{F}$. Then

1. $0\mathbf{a} = \mathbf{0}$ for all $\mathbf{a} \in V$,
2. $(-1)\mathbf{a} = -\mathbf{a}$ for all $\mathbf{a} \in V$,
3. $\mathbf{0} + \mathbf{a} = \mathbf{a}$ for all $\mathbf{a} \in V$, and
4. $r\mathbf{0} = \mathbf{0}$ for all $r \in \mathbb{F}$.