Vector Spaces (Continued)

Complex Numbers: Before proceeding any further, let's take a moment to look at the complex numbers and their properties.

Definition: A **complex number** is an order pair (a, b) where $a, b \in \mathbb{R}$. We write this as

We define addition and multiplication in $\mathbb{C}$ by the rules: if $z = a + bi$ and $w = c + di$, then

$$z + w = (a + c) + (b + d)i$$

and

$$z \cdot w = (a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

Some Properties of Complex Numbers: Let z and w be complex numbers. We have:

1. $z + \overline{z} = 2\text{Re}(z)$
2. $z\overline{z} = |z|^2$
3. $\overline{w + z} = \overline{w} + \overline{z}$ and $\overline{wz} = \overline{w} \cdot \overline{z}$
4. $\overline{\overline{z}} = z$
5. $|\text{Re}(z)| \leq |z|$ and $|\text{Im}(z)| \leq |z|$
6. $|\overline{z}| = |z|$
7. $|wz| = |w| \cdot |z|$
8. $|w + z|$

Example: Let

$$\mathbb{C}^n = \left\{ \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} : z_1, \ldots, z_n \in \mathbb{C} \right\}.$$

We define "vector addition" and "scalar multiplication" by the rules

With these operations, $\mathbb{C}^n$ is a

More Examples of Vector Spaces:

1. Let S be a non-empty set. Define

$$V = Fun(S) = \{\text{functions } f : S \to \mathbb{R}\}.$$

Given $f, g \in Fun(S)$ and $\lambda \in \mathbb{R}$, we define $f + g : S \to \mathbb{R}$ and $\lambda f : S \to \mathbb{R}$ by the rules

$$(f + g)(x) = f(x) + g(x)$$

and

$$(\lambda f)(x) = \lambda f(x).$$

2. Fix real numbers a and b such that $a < b$. We define

$$\mathcal{C}([a, b]) = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuous}\}.$$

For $f, g \in \mathcal{C}([a, b])$ and $\lambda \in \mathbb{R}$, we define $f + g : [a, b] \to \mathbb{R}$ and $\lambda f : [a, b] \to \mathbb{R}$ by the rules

$$(f + g)(x) = f(x) + g(x)$$

and

$$(\lambda f)(x) = \lambda f(x).$$

3. Let V be the set of all lines in $\mathbb{R}^2$ with slope 1. That is,

$$V = \{y = x + d \mid d \in \mathbb{R}\}.$$

Given $y_1 = x + d_1$ and $y_2 = x + d_2$ and $\lambda \in \mathbb{R}$, we define $y_1 + y_2$ and λy_1 as

$$y_1 + y_2 = x + (d_1 + d_2)$$

and

$$\lambda y_1 = x + \lambda d_1$$

4. Let V and W be two vector spaces over the same field $\mathbb{F}$. Let

$$V \oplus W = \{\mathbf{A}, \mathbf{B}) \mid \mathbf{A} \in V, \mathbf{B} \in W\}.$$

We define vector addition and scalar multiplication with scalars in $\mathbb{F}$ coordinate-wise:

$$(\mathbf{A}, \mathbf{B}) + (\mathbf{A}', \mathbf{B}') = (\mathbf{A} + \mathbf{A}', \mathbf{B} + \mathbf{B}')$$

and

$$\lambda(\mathbf{A}, \mathbf{B}) = (\lambda \mathbf{A}, \lambda \mathbf{B}).$$

Example of a Non-Vector Space: Consider $V = \{p(x) = a_0 + a_1 x \mid a_0, a_1 \in \mathbb{C}\}$. That is, V is the collection of all polynomials of degree at most 1 with coefficients in the complex numbers. We define addition and scalar multiplication operations (with scalars from $\mathbb{C}$) on V as follows:

$$(a_0 + a_1 x) + (b_0 + b_1 x) = (a_0 + a_1) + (b_0 + b_1)x$$

and

$$\alpha(a_0 + a_1 x) = \alpha a_1 + \alpha a_0 x.$$

Proposition: Let V be a vector space over a field $\mathbb{F}$. Then

1. $0\mathbf{a} = \mathbf{0}$ for all $\mathbf{a} \in V$,
2. $(-1)\mathbf{a} = -\mathbf{a}$ for all $\mathbf{a} \in V$,
3. $\mathbf{0} + \mathbf{a} = \mathbf{a}$ for all $\mathbf{a} \in V$,
4. $r\mathbf{0} = \mathbf{0}$ for all $r \in \mathbb{F}$,
5. the zero vector is unique, and
6. additive inverses are unique.