Symmetric And Hermitian Matrices

Definitions: Let A be a matrix in $M_{m\times n}(\mathbb{F})$.

1. The **adjoint** of A, denoted A^*, is the conjugate transpose of A. That is, $A^* = \overline{A}^T$.
2. A is **Hermitian** if $A = A^*$.
3. If $\mathbb{F} = \mathbb{R}$, then A is said to be **symmetric** if

Examples:

1. $A = \begin{bmatrix} 2 & 4 \\ -i & 2+i \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$

Proposition: Let $A, B \in M_{m\times n}(\mathbb{F})$. Then

1. $(AB)^* = B^*A^*$;
2. $(A^*)^* = A$;
3. If $\mathbb{F} = \mathbb{R}$ then $A^* = A^T$;
4. $(\alpha A)^* = \overline{\alpha} A^*$.

Definitions: Let $U \in M_{n\times n}(\mathbb{F})$.

1. U is said to be **unitary** if

2. If $\mathbb{F} = \mathbb{R}$, then we say that U is **orthogonal** if

Example: The matrix

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}$$

Proposition: Let $U \in M_{n\times n}(\mathbb{F})$ be unitary and equip $\mathbb{C}^n$ with the standard Hermitian inner product. Then

1. The rows of U form an orthonormal basis of $\mathbb{C}^n$.
2. The columns of U form an orthonormal basis of $\mathbb{C}^n$.
3.

Definitions: Let $A \in M_{n\times n}(\mathbb{F})$.

1. When $\mathbb{F} = \mathbb{R}$, we say that A is **orthogonally diagonalizable** if there is an orthogonal matrix P and a diagonal matrix D such that
$$D = P^{-1}AP = P^T AP.$$
2. When $\mathbb{F} = \mathbb{C}$, we say that A is **unitarily diagonalizable** if there is an unitary matrix P and a diagonal matrix D such that
$$D = P^{-1}AP = P^* AP.$$

Spectral Theorems: Let $A \in M_{n\times n}(\mathbb{F})$.

1. A is Hermitian if and only if it is

2. When $\mathbb{F} = \mathbb{R}$, A is symmetric if and only if it is

Example: The matrix

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Quadratic Forms

Goal: To study sums of squares.

Definition: A **quadratic form on** $\mathbb{R}^n$ is a function $Q : \mathbb{R}^n \to \mathbb{R}$ of the form

Examples:

1. The matrix $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

2. Let $\mathbf{x} \in \mathbb{R}^3$ and $Q(\mathbf{x}) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$. Write $Q(\mathbf{x})$ as $\mathbf{x}^T A\mathbf{x}$ for some symmetric matrix A.

3. Let $Q(\mathbf{x}) = x_1^2 - 8x_1x_2 - 5x_2^2$.

Note: Life is often easier when there are no cross-product terms!

Change of Variable: This is an equation of the form $\mathbf{x} = P\mathbf{y}$ for some invertible matrix P.

If $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ and we do a change of variable, then we have

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T(P^T AP)\mathbf{y}.$$

Example: The matrix

$$A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$$

is orthogonally diagonalizable (since it is symmetric).

Principal Axis Theorem: Let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix. Then there is an orthogonal change of variable $\mathbf{x} = P\mathbf{y}$ that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into a quadratic form $\mathbf{y}^T D\mathbf{y}$ with no cross-product terms.

Note: In $\mathbb{R}^2$, if $A \in M_{2\times 2}(\mathbb{R})$ is a symmetric matrix and we fix the real number c, then the set

$$L_c = \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x}^T A\mathbf{x} = c\}$$

is one of