Subspaces (Continued)

Fix real numbers a and b such that $a < b$. Last class we looked at the vector spaces $Fun([a, b])$ and $\mathcal{C}([a, b])$ – both are vector spaces over $\mathbb{R}$ and $\mathcal{C}([a, b])$ lives inside $Fun([a, b])$. Is there a relationship between these vector spaces? Let's look at some additional examples.

Motivating Examples:

1. Let $V = \mathbb{R}^3$ and
$$\mathcal{U} = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$
Note that $\mathcal{U} \subseteq V$. We define "vector addition" and "scalar multiplication" with scalars from $\mathbb{R}$ on $\mathcal{U}$ component-wise as we do for $V = \mathbb{R}^3$. One can check that $\mathcal{U}$ is a vector space over $\mathbb{R}$ as we do for $\mathbb{R}^3$.

2. Let
$$\mathcal{U} = \{p(x) \in \mathcal{P}_2(\mathbb{R}) : p(3) = 0\}.$$
We define "vector addition" and "scalar multiplication" with scalars from $\mathbb{R}$ as we do in $\mathcal{P}_2(\mathbb{R})$. Indeed, both

Definition: Let V be a vector space over the field $\mathbb{F}$ and let $\mathcal{U} \subseteq V$ be a subset of V. We call $\mathcal{U}$ a (**linear/vector**) **subspace** of V if

Remark: Checking the vector space axioms can be a lot of work! But checking them on $\mathcal{U}$ mostly comes for free from the properties of V being a vector space. In fact, we need only test 3 things!

Theorem (Subspace Test): Suppose $\mathcal{U}$ is a subset of a vector space V over the field $\mathbb{F}$. The subset $\mathcal{U}$ is a subspace of V if and only if the following three conditions holds:

(i) $\mathcal{U}$ is non-empty;

(ii) For all $\mathbf{x}, \mathbf{y} \in \mathcal{U}$, we have $\mathbf{x} + \mathbf{y}$

(iii) For all $\alpha \in \mathbb{F}$ and for all $\mathbf{x} \in \mathcal{U}$, we have $\alpha\mathbf{x}$

Proof: First suppose that $\mathcal{U}$ is a subspace of V. Then $\mathcal{U}$ has a zero vector and hence is non-empty. Also, properties (ii) and (iii) hold since $\mathcal{U}$ is a vector space and we have closure under vector addition and scalar multiplication for any vector space.

Conversely, assume that properties (i), (ii), and (iii) hold for the subset $\mathcal{U}$.

Examples:

1. The subset $\mathcal{U} = \{p(x) = a_0 + a_1x + a_2x^2 \in \mathcal{P}_2(\mathbb{R}) : p(3) = 0\}$ is a vector subspace of $\mathcal{P}_2(\mathbb{R})$ since

2. Let
$$L = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 : a, b \in \mathbb{Z} \right\}.$$

Question: How can we make subspaces? Let's look at two ways today.

I - Spanning Sets

Definitions: Let V be a vector space over the field $\mathbb{F}$. Let $\mathbf{A_1}, \ldots, \mathbf{A_n}$ be vectors in V.

1. A **linear combination** of $\mathbf{A_1}, \ldots, \mathbf{A_n}$ is

2. Write $\mathcal{B} = \{\mathbf{A_1}, \ldots, \mathbf{A_n}\}$. The (**linear**) **span** of $\mathcal{B}$ is the set of all linear combinations of $\{\mathbf{A_1}, \ldots, \mathbf{A_n}\}$, denoted

Examples:

1. Let
$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subseteq \mathbb{R}^3.$$

2. Using $\mathbb{F} = \mathbb{R}$, we have

3. In $\mathcal{P}_2(\mathbb{R})$,

4. Let
$$H = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\} \subseteq \mathbb{R}^4.$$

Proposition: Let $\mathcal{B} = \{\mathbf{A_1}, \ldots, \mathbf{A_n}\}$ be a subset of the vector space V over the field $\mathbb{F}$. Then $\mathcal{L}(\mathbf{A_1}, \ldots, \mathbf{A_n}) = Span(\mathbf{A_1}, \ldots, \mathbf{A_n}) = Span(\mathcal{B})$ is a subspace of V.

Proof:

II - Direct Sums

Definitions: Let $\mathcal{U}_1, \ldots, \mathcal{U}_m$ be subsets of the vector space V over the field $\mathbb{F}$.

1. We define the sum
$$\mathcal{U}_1 + \cdots + \mathcal{U}_m = \{\mathbf{u_1} + \cdots + \mathbf{u_m} \mid \mathbf{u_1} \in \mathcal{U}_1, \ldots, \mathbf{u_m} \in \mathcal{U}_m\}.$$

2. If $\mathcal{U}_1, \ldots, \mathcal{U}_m$ are subspaces of V then $\mathcal{U} = \mathcal{U}_1 + \cdots + \mathcal{U}_m$ is called a **direct sum**, denoted $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \cdots \oplus \mathcal{U}_m$, if each element of $\mathcal{U}_1 + \cdots + \mathcal{U}_m$ can be written in only one way as $\mathbf{u_1} + \cdots + \mathbf{u_m}$ with each $\mathbf{u_j}$ in $\mathcal{U}_j$.

Theorem: Let $\mathcal{U}_1, \ldots, \mathcal{U}_m$ be subspaces of the vector space V over the field $\mathbb{F}$.

1. $\mathcal{U}_1 + \cdots + \mathcal{U}_m$ is a subspace of V.
2. $\mathcal{U}_1 + \cdots + \mathcal{U}_m$ is a direct sum if and only if the only way to write the zero vector $\mathbf{0}$ as a sum of the form $\mathbf{u_1} + \cdots + \mathbf{u_m}$ with each $\mathbf{u_j}$ in $\mathcal{U}_j$ is to take each $\mathbf{u_j}$ to be $\mathbf{0}$.

Theorem: Let $\mathcal{U}$ and $\mathcal{W}$ be subspaces of the vector space V over the field $\mathbb{F}$. Then $V = \mathcal{U} \oplus \mathcal{W}$ if and only if $V = \mathcal{U} + \mathcal{W}$ and

$$\mathcal{U} \cap \mathcal{W} = \{\mathbf{x} \in V : \mathbf{x} \in \mathcal{U} \text{ and } \mathbf{x} \in \mathcal{W}\} = \{\mathbf{0}\}.$$

Example: Let $V = \mathbb{R}^3$,

$$\mathcal{U} = Span\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}\right)$$

and

$$\mathcal{W} = Span\left(\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}\right).$$