Matrices Of Linear Transformations

Motivating Example: Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y+z \end{bmatrix}.$$

- Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$
- Let $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Question: Can we represent linear transformations of "abstract" vector spaces by matrices?

Example: $T : \mathcal{P}_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(a + bx + cx^2) = \begin{bmatrix} a - 2b & 4c \\ a + b + c & b - c \end{bmatrix}$$

is a linear transformation.

- Let $\mathcal{B} = \{1, x, x^2\}$
- Let $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

T is completely determined by $T(1), T(x), T(x^2)$. Indeed, we have

Theorem: Let V be an n-dimensional vector space with ordered basis $\mathcal{B}$. Let W be an m-dimensional vector space with ordered basis $\mathcal{C}$. For any linear transformation $T : V \to W$ there exists an $m \times n$ matrix A such that

$$[T(\mathbf{v})]_{\mathcal{C}} = A[\mathbf{v}]_{\mathcal{B}}$$

for all $\mathbf{v}$ in V. Conversely, every $m \times n$ matrix A defines a linear transformation $T : V \to W$ by $[T(\mathbf{v})]_{\mathcal{C}} = A[\mathbf{v}]_{\mathcal{B}}$.

Corollary/Definition: Let V be a vector space with ordered basis $\mathcal{B} = \{\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n\}$. Let W be a vector space with ordered basis $\mathcal{C} = \{\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_m\}$. Let $T : V \to W$ be a linear transformation. The $m \times n$ matrix A such that

$$[T(\mathbf{v})]_{\mathcal{C}} = A[\mathbf{v}]_{\mathcal{B}}$$

for all $\mathbf{v}$ in V is given by

Fact: The linear transformation

$$\mathcal{L}(V,W) \to Mat_{m\times n}(\mathbb{F})$$

defined by

$$T \mapsto [T]_{\mathcal{B}}^{\mathcal{C}}$$

is an isomorphism, and so

Example: Let $D : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be the differentiation transformation. That is,

$$D(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2.$$

We use the standard bases:

- $\mathcal{B} = \{1, x, x^2, x^3\}$
- $\mathcal{C} = \{1, x, x^2\}$