Matrices Of Linear Transformations: Images and Kernels

Examples:

1. Let $D : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be the differentiation transformation. That is,

$$D(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2.$$

We use the standard bases:

- $\mathcal{B} = \{1, x, x^2, x^3\}$
- $\mathcal{C} = \{1, x, x^2\}$

2. Let $T : \mathbb{R}^4 \to \mathcal{P}_1(\mathbb{R})$ be the linear transformation defined by

$$T((a_1, a_2, a_3, a_4)) = (a_1 + a_3) + (a_2 + a_4)x.$$

We use the bases:

- $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ for $\mathbb{R}^4$
- $\mathcal{C} = \{1 + x, 1 - x\}$ for $\mathcal{P}_1(\mathbb{R})$

Why do we care to represent linear transformations by matrices? We can learn about linear transformations using the computational machinery of matrices! Let's investigate this with images and kernels.

Recall/Definition: Let A be an $m \times n$ matrix with entries in $\mathbb{F}$.

1. The **column space** of A is the span of the columns of A, denoted $Col(A)$.
2. The **nullspace** (or **kernel**) of A, denoted $Null(A)$, is

Note:

1. $Col(A)$ is a subspace of $\mathbb{F}^m$.
2. $Null(A)$ is a subspace of $\mathbb{F}^n$.

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 5 & -3 & -8 \\ -2 & -4 & -11 & 2 & 4 \\ -1 & -2 & -6 & -1 & -4 \\ 1 & 2 & 5 & -2 & -5 \end{bmatrix}.$$

Proposition: Let $T : V \to W$ be a linear transformation and $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W, respectively. Let $A = [T]_{\mathcal{B}}^{\mathcal{C}}$. Then

1. $\mathbf{v} \in \ker(T)$ if and only if $[\mathbf{v}]_{\mathcal{B}} \in Null(A)$.
2. $\mathbf{w} \in \mathrm{Im}(T)$ if and only if $[\mathbf{w}]_{\mathcal{C}} \in Col(A)$.

Example: Consider the linear transformation $T : \mathcal{P}_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(a + bx + cx^2) = \begin{bmatrix} a+b+c & a-b+3c \\ 3a+b+5c & 0 \end{bmatrix}.$$

Goal: To find bases for $\ker(T)$ and $\mathrm{Im}(T)$. We use the standard bases for our vector spaces:

- $\mathcal{B} = \{1, x, x^2\}$ for $\mathcal{P}_2(\mathbb{R})$
- $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ for $M_{2\times 2}(\mathbb{R})$