# Matrices Of Linear Transformations: Change Of Bases

**Question:** Why do we multiply matrices the way we do? It seems unmotivated and "silly".

**Answer:** So that matrix representations of linear transformations "behave nicely".

**Proposition:** Let $\mathcal{U}, V$ and $W$ be finite-dimensional vector spaces with bases $\mathcal{C}, \mathcal{D}, \mathcal{B}$, respectively. Let $S : \mathcal{U} \to V$, $T : V \to W$ and $G : V \to W$ be linear transformations and $\alpha$ be a scalar. Then

1. $[\alpha T]_{\mathcal{D}}^{\mathcal{B}} = \alpha [T]_{\mathcal{D}}^{\mathcal{B}}$.
2. $[T + G]_{\mathcal{D}}^{\mathcal{B}} = [T]_{\mathcal{D}}^{\mathcal{B}} + [G]_{\mathcal{D}}^{\mathcal{B}}$.
3.

**Example:** Let $S : \mathbb{R}^2 \to \mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformations such that

$$A = [S]_{\mathcal{C}}^{\mathcal{D}} = \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 2 \end{bmatrix}$$

and

$$B = [T]_{\mathcal{D}}^{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \\ 3 & 2 & 0 \end{bmatrix}$$

where $\mathcal{C}, \mathcal{D}, \mathcal{B}$ are the standard bases of $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4$, respectively. Thus,

**Corollary:** A linear transformation $T : V \to V$ is an isomorphism if and only if $[T]_{\mathcal{B}}^{\mathcal{C}}$ is invertible where $\mathcal{B}$ and $\mathcal{C}$ are *any* bases of $V$.

**Change of Bases**

**Question:** Are some bases better than others for our matrix representations? If so, how do we change bases to our benefit?

**Example:** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T((x, y, z)) = (y + z, x + z, x + y).$$

- Using the standard basis $\mathcal{B}$ for $\mathbb{R}^3$, we have

- Using the basis $\mathcal{C} = \{(1,1,1),(1,-1,0),(1,1,-2)\}$ for $\mathbb{R}^3$, we have

**Theorem:** Let $A$ and $B$ be $m \times n$ matrices, $V$ be an $n$-dimensional vector space, and $W$ be an $m$-dimensional vector space. Then $A$ and $B$ represent the linear transformation $T : V \to W$ relative to ordered pairs of bases if and only if there are invertible matrices $P$ and $Q$ such that

$$A = PBQ^{-1}.$$

**Example:** Let's return to the previous example with $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T((x, y, z)) = (y + z, x + z, x + y).$$

We had

- $\mathcal{B}$ is the standard basis for $\mathbb{R}^3$;
- the basis $\mathcal{C} = \{(1, 1, 1), (1, -1, 0), (1, 1, -2)\}$ for $\mathbb{R}^3$;
- $A = [T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$;
- $B = [T]_{\mathcal{C}}^{\mathcal{C}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

By our previous theorem, we have invertible matrices $P$ and $Q$ such that

$$A = PBQ^{-1}.$$