Matrices Of Linear Transformations: Change Of Bases

Examples:

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T((x, y)) = (x + 3y, 3x + y).$$

We use the following two bases for $\mathbb{R}^2$:

- $\mathcal{A} = \mathcal{C} = \mathcal{D} = \{(1, 1), (1, -1)\}$;
- $\mathcal{B} = \{(1, 0), (0, 1)\}$.

On Worksheet 5, you showed that

$$A = [T]_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 4 & -2 \\ 4 & 2 \end{bmatrix}$$

and

$$B = [T]_{\mathcal{C}}^{\mathcal{D}} = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

So, from last class, we know that there are invertible matrices P and Q such that

$$A = PBQ^{-1}.$$

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T((x, y, z)) = (3y + 4z, 3x, 4x).$$

We use the following two bases for $\mathbb{R}^3$:

- $\mathcal{A} = \mathcal{B} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$;
- $\mathcal{C} = \mathcal{D} = \{(0, 4, -3), (5, 3, 4), (5, -3, -4)\}$.

On Worksheet 5, you showed that

$$A = [T]_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 0 & 3 & 4 \\ 3 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

and

$$B = [T]_{\mathcal{C}}^{\mathcal{D}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$

So, from last class, we know that there are invertible matrices P and Q such that

$$A = PBQ^{-1}.$$

Question: How can we change basis coordinates without a linear transformation given?

Theorem: Let V be an n-dimensional vector space with two bases $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ and $\mathcal{C} = \{\mathbf{c_1}, \ldots, \mathbf{c_n}\}$. The matrix given by

Definition: $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from the theorem above is called the **change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$**.

Notes:

1. $P_{\mathcal{C}\leftarrow\mathcal{B}} = [I_V]_{\mathcal{B}}^{\mathcal{C}}$ where $I_V(\mathbf{v}) = \mathbf{v}$ for all vectors $\mathbf{v} \in V$.
2. $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible.

Examples:

1. Consider the two bases for $\mathbb{R}^2$:
    - $\mathcal{B} = \{(1, -3), (-2, 4)\}$;
    - $\mathcal{C} = \{(-7, 9), (-5, 7)\}$.

2. Consider the two bases for $\mathcal{P}_2(\mathbb{R})$:

 - $\mathcal{B} = \{1, x, x^2\}$;
 - $\mathcal{C} = \{1, 1 + x, 1 + x + x^2\}$.