Matrices Of Linear Transformations: A Nice Example

Example: Given an isomorphism, we can sometimes easily find its inverse. For example, consider the linear transformation $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ given by

$$T(a + bx + cx^2) = (a, b, c).$$

This is an isomorphism with inverse $T^{-1} : \mathbb{R}^3 \to \mathcal{P}_2(\mathbb{R})$ defined by

$$T^{-1}((a, b, c)) = a + bx + cx^2.$$

Indeed,

Question: What can we do if the inverse formula is not easy to find? Can we use matrix representations?

Proposition: Let $T : V \to W$ be an isomorphism. Let $\mathcal{B}$ be a basis for V and $\mathcal{C}$ be a basis for W. Then $[T]_{\mathcal{B}}^{\mathcal{C}}$ is invertible and

$$([T]_{\mathcal{B}}^{\mathcal{C}})^{-1} = [T^{-1}]_{\mathcal{C}}^{\mathcal{B}}.$$

Example: Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ be the isomorphism given by

$$T(p(x)) = (p(-1), p(0), p(1)).$$

Let $\mathcal{B}$ and $\mathcal{C}$ be the standard bases of $\mathcal{P}_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively. Then

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Review: Determinants & Inverses

Definition: Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and jth column.

1. The **determinant** of A is defined by the inductive formula:

2. $C_{ij} = (-1)^{i+j} \det(A_{ij})$ is called the **cofactor** of a_{ij}.

Fact: We can compute $\det(A)$ by **cofactor expansion** along any row or column. That is,

$$\begin{aligned}\det(A) &= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \\ &= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}\end{aligned}$$

for all $1 \leq i, j \leq n$.

Example: Let

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 3 & -1 \\ 0 & 1 & -2 \end{bmatrix}.$$

Properties of Determinants: Let $A, B \in M_{n\times n}(\mathbb{F})$.

1. If A has a row or column of zeroes, then $\det(A) = 0$.
2. If A has 2 equal rows or two equal columns, then $\det(A) = 0$.
3. $\det(A) = 0$ if and only if the columns of A are linearly dependent as vectors in $\mathbb{F}^n$.
4. $\det(A) = 0$ if and only if the rows of A are linearly dependent as vectors in $\mathbb{F}^n$.
5. If A is an upper or lower triangular matrix, then the determinant of A is the product of the diagonal entries.
6. $\det(A^T) = \det(A)$.
7. $\det(AB) = \det(A)\det(B)$
8. If B is obtained from A by interchanging two rows, then $\det(B) = -\det(A)$.
9.

Definition: An $n \times n$ matrix is **invertible** if there exists a matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I_n$$

where I_n is the $n \times n$ identity matrix. We call A^{-1} the **inverse** of A.

Methods To Find A Matrix Inverse:

Definition: For *any* matrix B, we define the **rank** of B to be the number of leading 1's in the reduced row echelon form of B.

Theorem: Let $A \in M_{n\times n}(\mathbb{F})$. The following are equivalent:

1. $\det(A) \neq 0$;
2. $\text{rank}(A) = n$;
3. A is

Fact: If $A \in M_{n\times n}(\mathbb{F})$ is invertible then A^{-1} is unique and

$$\det(A^{-1}) = (\det(A))^{-1}.$$

Thus, we can prove the implication (3) $\implies$ (1) in the above theorem easily: