Wrap-Up Worksheet

Some Sample Solutions

1. Let $\underline{a} = \begin{bmatrix} a \\ b \end{bmatrix}$, $X = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$

Then

$$\underline{a} = (X^T X)^{-1} X^T y$$

$$= \left(\begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

$$= \frac{1}{84} \begin{bmatrix} 142 & -22 \\ -22 & 4 \end{bmatrix} \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

$$= \frac{1}{84} \begin{bmatrix} 24 \\ 30 \end{bmatrix}$$

$$= \begin{bmatrix} 2/7 \\ 5/14 \end{bmatrix}$$

So $y = \frac{2}{7} + \frac{5}{14} x$ is the line of best fit.

2) We need orthonormal bases for each eigenspace.

We apply Gram-Schmidt to

$$\underline{v_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{v_2} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

with the dot product for the inner product

Let $\underline{w_1} = \underline{v_1}$

$$\underline{w_2} = \underline{v_2} - \frac{\underline{w_1} \cdot \underline{v_2}}{\|\underline{w_1}\|^2} \underline{w_1}$$

$$= \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

So
$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

is a collection of orthogonal eigenvectors. We now normalize each vector and make the normalized vectors the columns of P:

$$P = \begin{bmatrix} 1/2 & 0 & -1/\sqrt{2} & -1/2 \\ -1/2 & 1/\sqrt{2} & 0 & -1/2 \\ 1/2 & 0 & 1/\sqrt{2} & -1/2 \\ 1/2 & 1/\sqrt{2} & 0 & 1/2 \end{bmatrix}$$

$$D = \begin{bmatrix} -8 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

and $D = P^{-1}AP = P^{T}AP$

since P is an orthogonal matrix

3 ⓐ Let $\underline{x} \in \text{Null}(A)$. Then, by definition,

$$A\underline{x} = \underline{0}$$

Thus, $A^*A\underline{x} = A^*\underline{0} = \underline{0}$

so that $\underline{x} \in \text{Null}(A^*A)$

Therefore, $\text{Null}(A) \subseteq \text{Null}(A^*A)$

Conversely, let $\underline{x} \in \text{Null}(A^*A)$ so that

$$A^*A\underline{x} = \underline{0}$$

Then

$$0 = \underline{x}^* A^* A \underline{x} = (A\underline{x})^* A\underline{x} = \langle A\underline{x}, A\underline{x} \rangle$$

Thus, $A\underline{x} = \underline{0}$ ie $\underline{x} \in \text{Null}(A)$

Hence, $\text{Null}(A^*A) \subseteq \text{Null}(A)$

(b) We know that an $n \times n$ matrix is invertible if and only if $\text{Null}(A) = \{\underline{0}\}$

But $\text{Null}(A) = \{\underline{0}\}$ if and only if the columns of A are linearly independent by the Rank-Nullity Theorem

$$\left(\begin{aligned} n &= \dim(\text{Null}(A)) + \dim(\text{Col}(A)) \\ &= 0 + \dim(\text{Col}(A)) \end{aligned} \right)$$

(4) View vectors in $\mathbb{C}^n$ as column matrices. Then by the definition of our inner product

$$\langle \underline{x}, \underline{y} \rangle = \underline{y}^* \underline{x}$$

Since $\mathcal{U}$ is unitary, we have

$$\langle U\underline{x}, U\underline{y}\rangle = (U\underline{y})^* U\underline{x}$$
$$= \underline{y}^* U^* U\underline{x}$$
$$= \underline{y}^* (U^{-1}) U\underline{x}$$
$$= \underline{y}^* \underline{x}$$
$$= \langle \underline{x}, \underline{y}\rangle$$

5 $$\underline{x}^T A \underline{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4x_1 + 3x_2 \\ 3x_1 + 2x_2 + x_3 \\ x_2 + x_3 \end{bmatrix}$$

$$= 4x_1^2 + 3x_1x_2 + 3x_1x_2 + 2x_2^2 + x_2x_3 + x_2x_3 + x_3^2$$

$$= 4x_1^2 + 2x_2^2 + x_3^2 + 6x_1x_2 + 2x_2x_3$$

6) Let $Q(\underline{x}) = 8x_1^2 + 7x_2^2 - 3x_3^3 - 6x_1x_2 + 4x_1x_3 - 2x_2x_3$

The coefficients of x_1^2, x_2^2, x_3^2 go on the diagonal of the matrix A.
To make A symmetric the coefficient of $x_i x_j$ (for $i \neq j$) is split-up evenly between the (i,j) and (j,i) entries of A

eg The coefficient of x_1x_2 is -6

So $a_{12} = a_{21} = \frac{1}{2}(-6) = -3$

$$\therefore \quad A = \begin{bmatrix} 8 & -3 & 2 \\ -3 & 7 & -1 \\ 2 & -1 & -3 \end{bmatrix}$$

<u>Check:</u>

$Q(\underline{x}) = \underline{x}^T A \underline{x}$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 8 & -3 & 2 \\ -3 & 7 & -1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

7 The matrix of the quadratic form is

$$A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 - 25$$

$$= \lambda^2 - 2\lambda - 24$$

$$= (\lambda + 4)(\lambda - 6)$$

So A has eigenvalues $\lambda_1 = -4$, $\lambda_2 = 6$

$\underline{E_{\lambda_1}} = \text{Null}(A + 4I)$

$$[A+4I \mid \underline{0}] = \left[\begin{array}{cc|c} 5 & 5 & 0 \\ 5 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So E_{λ_1} has basis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

Note that a basis of E_{λ_1} with unit eigenvector is $\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$

$\underline{E_{\lambda_2}} = \text{Null}(A - 6I)$

$$[A-6I \mid \underline{0}] = \left[\begin{array}{cc|c} -5 & 5 & 0 \\ 5 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So E_{λ_2} has basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

A basis of E_{λ_2} with unit eigenvector is $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$

Thus,

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$ orthogonally

diagonalizes A

The change of variable $\underline{x} = Py$ produces the quadratic form

$$y^T(P^TAP)y = y^TDy = -4y_1^2 + 6y_2^2$$

where $D = \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix}$