# Wrap-Up Worksheet

Name and Student Number: ______________________________

Work the following exercises. *Explain all of your reasoning.* Remember to use good notation and full sentences.

1. Find real numbers $a$ and $b$ such that the equation $y = a + bx$ is the line of best fit for the data

| $x$ | 2 | 5 | 7 | 8 |
|---|---|---|---|---|
| $y$ | 1 | 2 | 3 | 3 |

2. The symmetric matrix

$$A = \begin{bmatrix} 1 & 3 & -3 & -3 \\ 3 & -3 & 3 & -1 \\ -3 & 3 & 1 & -3 \\ -3 & -1 & -3 & -3 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$$

has eigenvalues $\lambda_1 = -8$ and $\lambda_2 = -4$ and $\lambda_3 = 4$. The eigenspace $E_{\lambda_1}$ has basis

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\},$$

the eigenspace $E_{\lambda_2}$ has basis

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\},$$

and the eigenspace $E_{\lambda_3}$ has basis

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \right\}.$$

Orthogonally diagonalize $A$.

3. For this exercise, think of vectors in $\mathbb{C}^n$ as $n \times 1$ column matrices with complex entries. Also, equip $\mathbb{C}^n$ with the standard Hermitian inner product. Let $A$ be an $n \times n$ matrix with (possibly) complex entries.

   (a) Prove that $\mathrm{Null}(A^*A) = \mathrm{Null}(A)$. [Hint: Consider $||A\mathbf{x}||^2$].

(b) Prove that $A^*A$ is invertible if and only if the columns of $A$ are linearly independent.

4. Let $U$ be an $n \times n$ unitary matrix and consider $\mathbb{C}^n$ with respect to the standard Hermitian inner product. Show that

$$\langle U\mathbf{x}, U\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$.

5. For $\mathbf{x} \in \mathbb{R}^3$, compute the quadratic form $\mathbf{x}^T A\mathbf{x}$ where

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

6. Find the matrix of the quadratic form $8x_1^2 + 7x_2^2 - 3x_3^2 - 6x_1x_2 + 4x_1x_3 - 2x_2x_3$ where $\mathbf{x} \in \mathbb{R}^3$.

7. Make a change of variable $\mathbf{x} = P\mathbf{y}$ that transforms the quadratic equation $x_1^2 + 10x_1x_2 + x_2^2$ into a quadratic form with no cross-product term. Find both $P$ and the new quadratic form.