Tutorial Worksheet #9
Some Sample Solutions

(1) We first start with the standard basis $\mathcal{B} = \{1, x, x^2\}$ of $P_2(\mathbb{C})$

Then
$$A = [T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$$

We now diagonalize A:

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -4 & -2 \\ -4 & 5-\lambda & -2 \\ -2 & -2 & 8-\lambda \end{vmatrix}$$

$$\underset{R_1 \mapsto R_1 - R_2}{=} \begin{vmatrix} 9-\lambda & \lambda-9 & 0 \\ -4 & 5-\lambda & -2 \\ -2 & -2 & 8-\lambda \end{vmatrix}$$

$$= (9-\lambda) \begin{vmatrix} 1 & -1 & 0 \\ -4 & 5-\lambda & -2 \\ -2 & -2 & 8-\lambda \end{vmatrix}$$

$$\underset{\substack{R_2 \mapsto R_2 + 4R_1 \\ R_3 \mapsto R_3 + 2R_1}}{=} (9-\lambda) \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1-\lambda & -2 \\ 0 & -4 & 8-\lambda \end{vmatrix}$$

$$= (9-\lambda) \begin{vmatrix} 1-\lambda & -2 \\ -4 & 8-\lambda \end{vmatrix}$$

$$= (9-\lambda)(\lambda^2 - 9\lambda)$$
$$= -\lambda(9-\lambda)^2$$

So the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 9$

$\underline{E_{\lambda_1}} = \text{Null}(A - 0I)$

$$[A - 0I \mid \underline{0}] = \left[\begin{array}{ccc|c} 5 & -4 & -2 & 0 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So E_{λ_1} has basis $\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$

$\underline{E_{\lambda_2}} = \text{Null}(A - 9I)$

$$[A - 9I \mid \underline{0}] = \left[\begin{array}{ccc|c} -4 & -4 & -2 & 0 \\ -4 & -4 & -2 & 0 \\ -2 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So E_{λ_2} has basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$

The set consisting of the basis vectors for the bases of E_{λ_1} and E_{λ_2} is a basis of $\mathbb{C}^3$ consisting of eigenvectors of A. Thus, converting back to $P_2(\mathbb{C})$,

$\mathcal{B} = \{2+2x+x^2, -1+x, -1+2x^2\}$ is a basis of eigenvectors of T for $P_2(\mathbb{C})$. Moreover,

$$D = [T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

2 (a) We have

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & 3 & -8 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & -3-\lambda \end{vmatrix}$$

$$= (-2-\lambda) \begin{vmatrix} 6-\lambda & -8 \\ 1 & -3-\lambda \end{vmatrix}$$

$$= -(\lambda - 5)(\lambda + 2)^2$$

Thus, for A to be diagonalizable we would need

geometric multiplicity of $\lambda = -2$ to be 2

Now $E_\lambda = \text{Null}(A + 2I)$

$$[A + 2I \mid \underline{0}] = \left[\begin{array}{ccc|c} 8 & 3 & -8 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that there is only 1 free variable and so

$$\underbrace{\dim(E_\lambda)}_{\substack{\| \\ \text{geometric multiplicity of } \lambda = -2}} = 1 \neq 2$$

Thus, A is not diagonalizable

(b) We have

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 0 \\ 3 & 1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix}$$

$$= (-2-\lambda) \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix}$$

$$= (-2-\lambda)(\lambda^2 - 2\lambda - 8)$$

$$= -(\lambda+2)^2(\lambda-4)$$

So A has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$

$\underline{E_{\lambda_1}} = \text{Null}(A+2I)$

$$[A+2I \mid \underline{0}] = \left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So E_{λ_1} has basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\underline{\text{ie}}$ $\dim(E_{\lambda_1}) = 2 =$ algebraic multiplicity of $\lambda_1 = -2$

$\underline{E_{\lambda_2}} = \text{Null}(A-4I)$

$$[A-4I \mid \underline{0}] = \left[\begin{array}{ccc|c} -3 & 3 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So E_{λ_2} has basis $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$

ie $\dim(E_{\lambda_2}) = 1 =$ algebraic multiplicity of $\lambda_2 = 4$

We conclude that A is diagonalizable.
with
$$P = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

3 @ Note that A is a diagonal matrix. Thus, if B and C are not diagonalizable then they cannot be similar to A.

We have
$$\det(B - \lambda I) = \det(C - \lambda I) = (2-\lambda)^3$$
So $\lambda = 2$ is only e-value for B & C.

__For B:__

$E_\lambda = \text{Null}(B - 2I)$

$$[B-2I \mid \underline{0}] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, $\dim(E_\lambda) = 2 =$ geometric multiplicity of λ

$< 3 =$ algebraic multiplicity of λ

and so B is not diagonalizable and hence not similar to A.

<u>For C:</u>

$E_\lambda = \text{Null}(C-2I)$

$$[C-2I \mid \underline{0}] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here, $\lambda = 2$ has geometric multiplicity equal to 1 which is again less than $3 =$ algebraic multiplicity of λ

We conclude that C is not diagonalizable and so not similar to A.

(b) Since S and T are similar, there exists an invertible matrix P such that $T = P^{-1}SP$.
Thus,

$$a_n T^n + \cdots + a_1 T + a_0 I$$

$$= a_n (P^{-1}SP)^n + \cdots + a_1 (P^{-1}SP) + a_0 P^{-1}P$$

$$= a_n P^{-1}S^n P + \cdots + a_1 P^{-1}SP + a_0 P^{-1}P$$

$$= P^{-1}(a_n S^n + \cdots + a_1 S + a_0 I)P$$

$$= P^{-1} \underline{0} P$$

$$= \underline{0} \quad \boxtimes$$

(c) $B^2 - 4B + 4I$

$$= \begin{bmatrix} 4 & 4 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 8 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

d By part b, it suffices to show

$$C^2 - 4C + 4I \neq \underline{0}$$

We have

$$C^2 - 4C + 4I = \begin{bmatrix} 4 & 4 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 8 & 4 & 0 \\ 0 & 8 & 4 \\ 0 & 0 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \underline{0}$$

Thus, B is not similar to C.

4 a $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= -\lambda(1-\lambda) - 1$$

$$= \lambda^2 - \lambda - 1$$

So $\det(A - \lambda I) = \lambda^2 - \lambda - 1 = 0$ when

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

So A has eigenvalues $\lambda_1 = \frac{1+\sqrt{5}}{2}$

and $\lambda_2 = \frac{1-\sqrt{5}}{2}$

Now $\underline{E_{\lambda_1}} = \mathrm{Null}(A - \lambda_1 I)$

<u>Note</u> $\lambda_1^{-1} = -\lambda_2$, $\lambda_2^{-1} = -\lambda_1$

and

$$1 - \lambda_1 = 1 - \frac{1+\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} = \lambda_2$$

So

$$[A - \lambda_1 I \mid \underline{0}] = \left[\begin{array}{cc|c} -\lambda_1 & 1 & 0 \\ 1 & 1-\lambda_1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} \lambda_2^{-1} & 1 & 0 \\ 1 & \lambda_2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & \lambda_2 & 0 \\ 1 & \lambda_2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So a basis for E_{λ_1} is $\left\{ \begin{bmatrix} -\lambda_2 \\ 1 \end{bmatrix} \right\}$

Similarly, a basis for E_{λ_2} is $\left\{ \begin{bmatrix} -\lambda_1 \\ 1 \end{bmatrix} \right\}$

We use $P = \begin{bmatrix} -\lambda_2 & -\lambda_1 \\ 1 & 1 \end{bmatrix}$

and

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

so that $P^{-1} A P = D$

(b)

$$A^n = (PDP^{-1})^n$$

$$= P D^n P^{-1}$$

Now $P^{-1} = \dfrac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & \lambda_1 \\ -1 & -\lambda_2 \end{bmatrix}$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \lambda_1 \\ -1 & -\lambda_2 \end{bmatrix}$$

So

$$A^n = \frac{1}{\sqrt{5}} \begin{bmatrix} -\lambda_2 & -\lambda_1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 \\ -1 & -\lambda_2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n-1} & \lambda_2^{n-1} \\ \lambda_1^n & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 \\ -1 & -\lambda_2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n-1} - \lambda_2^{n-1} & \lambda_1^n - \lambda_2^n \\ \lambda_1^n - \lambda_2^n & \lambda_1^{n+1} - \lambda_2^{n+1} \end{bmatrix}$$

(c) $A^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}$

So by (b) we have

$$F_n = \frac{1}{\sqrt{5}} \left(\lambda_1^n - \lambda_2^n \right)$$

$$= \frac{1}{\sqrt{5}} \left(\frac{(1+\sqrt{5})^n}{2^n} - \frac{(1-\sqrt{5})^n}{2^n} \right)$$

$$= \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \sqrt{5}}$$

5 Let $\underline{v} = \begin{pmatrix} i \\ i \end{pmatrix} \in \mathbb{C}^2$

Then

$$\langle \underline{v}, \underline{v} \rangle = i(i) + i(i) = -1-1 = -2 < 0$$

So we do not have an inner product here

6 (a) $\langle \underline{u}, \underline{v} \rangle = (i)\overline{(-i)} + (1+i)\overline{(3-i)}$

$= i(i) + (1+i)(3+i)$

$= i^2 + 3 + 4i + i^2$

$= 1 + 4i$

(b) $\langle \underline{u}, \underline{v} \rangle = (1+i)\overline{(1+i)} + (-i)\overline{(-i)} + 2\overline{(2)}$

$= (1+i)(1-i) + (-i)(i) + 2(2)$

$= 1 - i^2 - i^2 + 4$

$= 7$

7 (a) This is an inner product! We verify this via our definition of inner product:

Let $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \in \mathbb{C}^3$

and $\alpha \in \mathbb{C}$. Then

(i) $$\langle \underline{v}, \underline{w} \rangle = \overline{v_1 \overline{w_1} + 2v_2\overline{w_2} + 3v_3\overline{w_3}}$$
$$= \overline{w_1\overline{v_1}} + 2w_2\overline{v_2} + 3w_3\overline{v_3}$$
$$= \langle \underline{w}, \underline{v} \rangle$$

(ii) $$\langle \alpha \underline{v}, \underline{w} \rangle = \alpha v_1 \overline{w_1} + 2\alpha v_2 \overline{w_2} + 3\alpha v_3 \overline{w_3}$$
$$= \alpha \left(v_1\overline{w_1} + 2v_2\overline{w_2} + 3v_3\overline{w_3} \right)$$
$$= \alpha \langle \underline{v}, \underline{w} \rangle$$

(iii) $$\langle \underline{v} + \underline{u}, \underline{w} \rangle = (v_1+u_1)\overline{w_1} + 2(v_2+u_2)\overline{w_2} + 3(v_3+u_3)\overline{w_3}$$
$$= \left(v_1\overline{w_1} + 2v_2\overline{w_2} + 3v_3\overline{w_3} \right)$$
$$+ \left(u_1\overline{w_1} + 2u_2\overline{w_2} + 3u_3\overline{w_3} \right)$$
$$= \langle \underline{v}, \underline{w} \rangle + \langle \underline{u}, \underline{w} \rangle$$

iv (a)

$$\langle \underline{v}, \underline{v} \rangle = v_1\bar{v}_1 + 2v_2\bar{v}_2 + 3v_3\bar{v}_3$$
$$= |v_1|^2 + 2|v_2|^2 + 3|v_3|^2 \geq 0$$

(b) If $\underline{v} = \underline{0}$ then clearly $\langle \underline{v}, \underline{v} \rangle = 0$

Conversely, if $\langle \underline{v}, \underline{v} \rangle = 0$ then

$$|v_1|^2 + 2|v_2|^2 + 3|v_3|^2 = 0$$

and so $|v_1| = |v_2| = |v_3| = 0$

and hence $v_1 = v_2 = v_3 = 0$ ie $\underline{v} = \underline{0}$.

(b) This is not an inner product

For example, let $\underline{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Then

$$\langle 2\underline{v}, \underline{v} \rangle = 2+1+2+1 = 6 \neq 8 = 2(1+1+1+1)$$
$$= 2\langle \underline{v}, \underline{v} \rangle$$

8 Let $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \in \mathbb{R}^3$

and $\alpha \in \mathbb{R}$. Then

i

$$\langle \underline{v}, \underline{w} \rangle = 2v_1w_1 + 2v_2w_2 + 2v_3w_3 - v_1w_2 - v_2w_1 - v_2w_3 - v_3w_2$$

$$= \underline{2w_1v_1 + 2w_2v_2} + 2w_3v_3 - w_1v_2 - w_2v_1 - w_2v_3 - w_3v_2$$

$$= \langle \underline{w}, \underline{v} \rangle \quad \text{(since we are in } \mathbb{R}\text{)}$$

ii $\langle \alpha\underline{v}, \underline{w} \rangle = 2\alpha v_1w_1 + 2\alpha v_2w_2 + 2\alpha v_3w_3 - \alpha v_1w_2 - \alpha v_2w_1 - \alpha v_2w_3 - \alpha v_3w_2$

$$= \alpha(2v_1w_1 + 2v_2w_2 + 2v_3w_3 - v_1w_2 - v_2w_1 - v_2w_3 - v_3w_2)$$

$$= \alpha \langle \underline{v}, \underline{w} \rangle$$

iii $\langle \underline{v} + \underline{u}, \underline{w} \rangle$

$$= 2(v_1+u_1)w_1 + 2(v_2+u_2)w_2 + 2(v_3+u_3)w_3 - (v_1+u_1)w_2 - (v_2+u_2)w_1 - (v_2+u_2)w_3 - (v_3+u_3)w_2$$

$= 2v_1w_1 + 2v_2w_2 + 2v_3w_3 - v_1w_2 - v_2w_1 - v_2w_3 - v_3w_2$

$\quad + 2u_1w_1 + 2u_2w_2 + 2u_3w_3 - u_1w_2 - u_2w_1 - u_2w_3 - u_3w_2$

$= \langle \underline{v}, \underline{w} \rangle + \langle \underline{u}, \underline{w} \rangle$

iv ⓐ $\langle \underline{v}, \underline{v} \rangle = 2v_1^2 + 2v_2^2 + 2v_3^2 - 2v_1v_2 - 2v_2v_3$

$= v_1^2 + v_1^2 - 2v_1v_2 + v_2^2 + v_2^2 - 2v_2v_3$

$\quad + v_3^2 + v_3^2$

$= v_1^2 + (v_1 - v_2)^2 + (v_2 - v_3)^2 + v_3^2$

≥ 0

ⓑ Clearly $\underline{v} = \underline{0} \Rightarrow \langle \underline{v}, \underline{v} \rangle = 0$

Also, if $\langle \underline{v}, \underline{v} \rangle = 0$ then by ⓐ we must have $v_1 = v_1 - v_2 = v_2 - v_3 = v_3 = 0$ and thus $v_1 = v_2 = v_3 = 0$ so that $\underline{v} = \underline{0}$

So, by definition, this is an inner product.

9 Since this is an inner product, we have

$\langle 2-4x+2x^2, p\rangle$

$= 2\langle 1,p\rangle - 4\langle x,p\rangle + 2\langle x^2,p\rangle$

$= 2(3) - 4(0) + 2(-1)$

$= 4$

10 (a) $\langle \underline{0}, \underline{v}\rangle = \langle 0(\underline{0}), \underline{v}\rangle$

$= 0\langle \underline{0}, \underline{v}\rangle \in \mathbb{F}$

$= 0$

(b) $\langle \underline{v}, \alpha\underline{w}\rangle = \overline{\langle \alpha\underline{w}, \underline{v}\rangle}$

$= \overline{\alpha\langle \underline{w}, \underline{v}\rangle}$

$= \bar{\alpha}\,\overline{\langle \underline{w}, \underline{v}\rangle}$

$= \bar{\alpha}\langle \underline{v}, \underline{w}\rangle$

c) $\|\alpha \underline{v}\| = \sqrt{\langle \alpha \underline{v}, \alpha \underline{v} \rangle}$

$$= \sqrt{\alpha \langle \underline{v}, \alpha \underline{v} \rangle}$$

$$= \sqrt{\alpha \bar{\alpha} \langle \underline{v}, \underline{v} \rangle}$$

$$= \sqrt{|\alpha|^2 \langle \underline{v}, \underline{v} \rangle}$$

$$= |\alpha| \sqrt{\langle \underline{v}, \underline{v} \rangle}$$

$$= |\alpha| \, \|\underline{v}\|$$

d) If $\|\underline{v}\| = 0$ then $\sqrt{\langle \underline{v}, \underline{v} \rangle} = 0$

and so $\langle \underline{v}, \underline{v} \rangle = 0^2 = 0$

But, by definition of inner product,

$\langle \underline{v}, \underline{v} \rangle = 0$ if and only if $\underline{v} = \underline{0}$

Hence, it must be that $\underline{v} = \underline{0}$

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(a) $\left\langle \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right\rangle$

$= tr\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right)$

$= tr\left(\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right)$

$= tr\left(\begin{bmatrix} 10 & 6 \\ 16 & 10 \end{bmatrix} \right)$

$= 10 + 10$

$= 20$

(b) $\left\| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\| = \sqrt{\left\langle \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle}$

$= \sqrt{tr\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)}$

$= \sqrt{tr\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)}$

$= \sqrt{tr\left(\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \right)}$

$= \sqrt{(2+2)} = \sqrt{4} = 2$

12 Let $\alpha \in \mathbb{F}$ and $\underline{x}, y \in V$. We have

i $T(\underline{x}+y) = \langle \underline{x}+y, \underline{v} \rangle$

$= \langle \underline{x}, \underline{v} \rangle + \langle y, \underline{v} \rangle$

$= T(\underline{x}) + T(y)$

and

ii $T(\alpha \underline{x}) = \langle \alpha \underline{x}, \underline{v} \rangle$

$= \alpha \langle \underline{x}, \underline{v} \rangle$

$= \alpha T(\underline{x})$

So, by definition, T is a linear transformation.