MATH 2090, Fall 2018 **Dr. S. Cooper**

Tutorial Worksheet #9

Monday, November 26

Name and Student Number: ______________________________

Work the following exercises. *Explain all of your reasoning.* Remember to use good notation and full sentences. All cited exercises are from the textbook *Linear Algebra* by Larry Smith.

1. Consider the linear transformation $T : \mathcal{P}_2(\mathbb{C}) \to \mathcal{P}_2(\mathbb{C})$ given by

$$T(a + bx + cx^2) = 5a - 4b - 2c + (-4a + 5b - 2c)x + (-2a - 2b + 8c)x^2.$$

Find a basis $\mathcal{B}$ of $\mathcal{P}_2(\mathbb{C})$ such that $[T]_{\mathcal{B}}^{\mathcal{B}} = D$, where D is a diagonal matrix. Write down the matrix D in this case.

2. Determine if the following matrices are diagonalizable. If the matrix is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(a) $A = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix} \in M_{3\times 3}(\mathbb{C})$

(b) $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \in M_{3\times 3}(\mathbb{C})$

3. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) Prove that A is not similar to B. Prove that A is not similar to C.

(b) Suppose S and T are similar matrices, and $a_n S^n + \cdots + a_1 S + a_0 I = \mathbf{0}$ where $a_i \in \mathbb{C}$ and $\mathbf{0}$ is the zero matrix. Prove that $a_n T^n + \cdots + a_1 T + a_0 I = \mathbf{0}$.

(c) Compute $B^2 - 4B + 4I$.

(d) Prove that B is not similar to C.

4. An application of taking powers of matrices: We will use diagonalization to find a closed form for the Fibonacci sequence. Recall that the Fibonacci sequence is the sequence defined by $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Writing out the first few terms, the sequence looks like $1, 1, 2, 3, 5, 8, 13, 21, \ldots$.

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Computing the first few powers of A we see

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad A^4 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, \quad \cdots \quad , \quad A^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}.$$

You do not need to prove the last equality, but it's true!

(a) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. The following fact will make your computations easier: If $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$, then $\lambda_1^{-1} = -\lambda_2$ and $\lambda_2^{-1} = -\lambda_1$.

(b) Compute A^n in terms of λ_1, λ_2, and n.

(c) Prove that

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

Note: You have now shown that the formula for F_n is always an integer. Moreover, you can compute the 1000th Fibonacci number without computing the first 999!

5. We mentioned in class that the usual dot product of vectors in $\mathbb{R}^2$ is an inner product on $\mathbb{R}^2$. Show that this is not the case if we replace $\mathbb{R}$ with $\mathbb{C}$. That is, if $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{C}^2$ then show that

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2$$

does not define an inner product on $\mathbb{C}^2$.

6. Find the following standard Hermitian inner products $\langle \mathbf{u}, \mathbf{v} \rangle$ for:

(a) $\mathbf{u} = (i, 1+i)$ and $\mathbf{v} = (-i, 3-i)$ in $\mathbb{C}^2$

(b) $\mathbf{u} = (1+i, -i, 2)$ and $\mathbf{v} = (1+i, -i, 2)$ in $\mathbb{C}^3$

7. For each of the the following two potential inner products on $\mathbb{C}^3$, decide whether or not they are inner products and justify your answer. Let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ be vectors in $\mathbb{C}^3$.

(a) $\langle \mathbf{v}, \mathbf{w} \rangle = v_1\overline{w_1} + 2v_2\overline{w_2} + 3v_3\overline{w_3}$.

(b) $\langle \mathbf{v}, \mathbf{w} \rangle = v_1 + \overline{w_1} + v_3 + \overline{w_3}$.

8. Prove that

$$\left\langle \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right\rangle = 2v_1w_1 + 2v_2w_2 + 2v_3w_3 - v_1w_2 - v_2w_1 - v_2w_3 - v_3w_2$$

is an inner product on $\mathbb{R}^3$.

9. In $\mathcal{P}_2(\mathbb{R})$, suppose $\langle 1, p \rangle = 3$, $\langle x, p \rangle = 0$ and $\langle x^2, p \rangle = -1$ for some $p \in \mathcal{P}_2(\mathbb{R})$. What is $\langle 2-4x+2x^2, p \rangle$?

10. Let V be an inner product space. Prove the following for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha \in \mathbb{F}$.

(a) $\langle \mathbf{0}, \mathbf{v} \rangle = 0$.

(b) $\langle \mathbf{v}, \alpha\mathbf{w} \rangle = \overline{\alpha}\langle \mathbf{v}, \mathbf{w} \rangle$.

(c) $||\alpha\mathbf{v}|| = |\alpha|||\mathbf{v}||$.

(d) If $||\mathbf{v}|| = 0$ then $\mathbf{v} = \mathbf{0}$.

11. Consider the vector space $M_{3\times 3}(\mathbb{R})$ and consider the inner product

$$\langle A, B \rangle = \mathrm{tr}(A^T B)$$

where tr denotes the trace of a matrix.

(a) Compute $\left\langle \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right\rangle$.

(b) What is the norm of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$?

12. Let V be an inner product space, and fix a vector $\mathbf{v} \in V$. Define a function $T : V \to \mathbb{F}$ by $T(\mathbf{w}) = \langle \mathbf{w}, \mathbf{v} \rangle$. Prove that T is a linear transformation.