Tutorial Worksheet # 8
Some Sample Solutions

1. Recall that A has size 3×3 and $\det(A) \neq 0$ if and only if $\operatorname{rank}(A) = 3$. Thus, since $\det(A) = 0$ we must have $\operatorname{rank}(A) < 3$.
Hence, $\operatorname{rank}(T) = \dim(\operatorname{Im}(T)) < 3 = \dim(W)$ and so $\operatorname{Im}(T) \neq W$. Thus, T is not surjective. By Exercise 7 on the "Thanksgiving Worksheet", T is also not injective.

2. $$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 4 & -5 \\ -1 & -5-\lambda & 7 \\ 0 & -2 & 3-\lambda \end{vmatrix}$$

$$\underset{(R_1 \to R_1 + R_2 - R_3)}{=} \begin{vmatrix} 1-\lambda & 1-\lambda & -(1-\lambda) \\ -1 & -5-\lambda & 7 \\ 0 & -2 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 1 & 1 & -1 \\ -1 & -5-\lambda & 7 \\ 0 & -2 & 3-\lambda \end{vmatrix}$$

$$\underset{(R_2 \to R_2 + R_1)}{=} (1-\lambda) \begin{vmatrix} 1 & 1 & -1 \\ 0 & -4-\lambda & 6 \\ 0 & -2 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -4-\lambda & 6 \\ -2 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(-4-\lambda)(3-\lambda)+12\right]$$

$$= (1-\lambda)(\lambda+\lambda^2)$$

$$= (1-\lambda)\lambda(1+\lambda)$$

So the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 0$ and $\lambda_3 = -1$

3) $E_\lambda = \text{Null}(A-2I)$

Now $\left[A-2I \mid \underline{0}\right] = \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array}\right]$

$$\sim \begin{array}{c} \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \end{array}$$

Let $x_3 = t$ and $x_2 = s$. Then

$$2x_1 = x_2 - 6x_3 = s - 6t \Rightarrow x_1 = s/2 - 3t$$

That is, if $(A-2I)\underline{x} = \underline{0}$ then

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/2 - 3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

So a basis for E_{λ_2} is

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

4) a) $AB = \begin{bmatrix} -2+i & 1+2i \\ -3-2i & -2+3i \end{bmatrix}$

$\Rightarrow tr(AB) = -2+i-2+3i = -4+4i$

$BA = \begin{bmatrix} -2+i & 3+2i \\ -1-2i & -2+3i \end{bmatrix}$

$\Rightarrow tr(BA) = -2+i-2+3i = -4+4i$

b) Let $A = [a_{ij}]$, $B = [b_{ij}]$ and denote the i,j entry of AB by $(AB)_{ij}$.

Then by definition of matrix multiplication and by regrouping terms, we have

$$\begin{aligned}
tr(AB) &= (AB)_{1,1} + (AB)_{2,2} + \cdots + (AB)_{n,n} \\
&= (\underline{a_{11}b_{11}} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}) \\
&+ (\underline{a_{21}b_{12}} + a_{22}b_{22} + \cdots + a_{2n}b_{n2}) \\
&+ \cdots + (\underline{a_{n1}b_{1n}} + a_{n2}b_{2n} + \cdots + a_{nn}b_{nn})
\end{aligned}$$

$$\begin{aligned}
&= \underline{(b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1n}a_{n1})} \\
&\quad + (b_{21}a_{12} + b_{22}a_{22} + \cdots + b_{2n}a_{n2}) \\
&\quad + \cdots + (b_{n1}a_{1n} + b_{n2}a_{2n} + \cdots + b_{nn}a_{nn}) \\
&= (BA)_{1,1} + (BA)_{2,2} + \cdots + (BA)_{n,n} \\
&= tr(BA)
\end{aligned}$$

c If A and B are similar then there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. Hence, by b, we have

$$\begin{aligned}
tr(B) = tr(P^{-1}AP) &= tr((P^{-1}A)P) \\
&= tr(P(P^{-1}A)) \\
&= tr((PP^{-1})A)
\end{aligned}$$

$= tr(IA)$

$= tr(A)$ $\boxtimes$

5 Let $A, B, C \in M_{n \times n}(\mathbb{F})$ such that A is similar to B and B is similar to C. Then, by definition, there exist invertible matrices P and Q such that

$$P^{-1}AP = B$$
$$Q^{-1}BQ = C$$

Hence,

$$C = Q^{-1}BQ = Q^{-1}(P^{-1}AP)Q$$
$$= (Q^{-1}P^{-1})A(PQ)$$
$$= (PQ)^{-1}A(PQ)$$

Since PQ is an invertible matrix, A is similar to C by definition.

6 We have:
B is similar to A and
A is similar to D.
Thus, by Exercise 5, B is similar to D. So, since D is a diagonal matrix, B is diagonalizable with diagonal matrix D.

7 a The characteristic polynomial of A is

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda-1)(\lambda+1)$$

Thus, the 2×2 matrix A has 2 distinct eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ and hence is diagonalizable.

Let's find bases for the eigenspaces.

$\underline{E_{\lambda_1}}$ = Null$(A - I)$

Now
$$[A - I \mid \underline{0}] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So E_{λ_1} has basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$E_{\lambda_2} = \text{Null}(A-(-1)I) = \text{Null}(A+I)$

Now

$$[A+I\,|\,\underline{0}] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So E_{λ_2} has basis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

We have $P^{-1}AP = D$ where

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(b) The characteristic polynomial of A is

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} - 3\begin{vmatrix} -3 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} -3 & -5-\lambda \\ 3 & 3 \end{vmatrix}$$

$$= (1-\lambda)(\lambda^2+4\lambda+4) - 3(3\lambda+6) + 3(3\lambda+6)$$
$$= (1-\lambda)(\lambda+2)^2 - 9(2+\lambda) + 9(\lambda+2)$$

$$= (\lambda+2)\left[(1-\lambda)(\lambda+2) - 9 + 9\right]$$

$$= (\lambda+2)^2(1-\lambda)$$

Thus, the eigenvalues of A are
$\lambda_1 = -2$ and $\lambda_2 = 1$

We now find bases for the eigenspaces.

$\underline{E_{\lambda_1}} = \text{Null}(A-(-2)I) = \text{Null}(A+2I)$

Now

$$[A+2I \mid \underline{0}] = \left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So a basis for E_{λ_1} is

$$\left\{ \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\underline{v}_1}, \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\underline{v}_2} \right\}$$

$\underline{E_{\lambda_2}} = \text{Null}(A-I)$

Now

$$[A-I \mid \underline{0}] = \left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So a basis for E_{λ_2} is

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\underline{v}_3} \right\}$$

Now it is straightforward to check that $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ is linearly independent and hence a basis for $\mathbb{F}^3$

Thus, $\mathbb{F}^3$ has a basis consisting of eigenvectors of A and hence A is diagonalizable.

We have $P^{-1}AP = D$ where

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8 (a) The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 0.4-\lambda & 0.6 \\ 0.5 & 0.5-\lambda \end{vmatrix}$$

$$= \left(\frac{4}{10} - \lambda\right)\left(\frac{5}{10} - \lambda\right) - \frac{30}{100}$$

$$= \frac{2}{10} - \frac{9}{10}\lambda + \lambda^2 - \frac{3}{10}$$

$$= \lambda^2 - \frac{9}{10}\lambda - \frac{1}{10}$$

$$= \frac{1}{10}(10\lambda^2 - 9\lambda - 1)$$

$$= \frac{1}{10}(10\lambda + 1)(\lambda - 1)$$

So the eigenvalues of A are
$\lambda_1 = -\frac{1}{10}$ and $\lambda_2 = 1$

We now find bases for the eigenspaces:

$\underline{E_{\lambda_1}} = \text{Null}\left(A - \left(-\frac{1}{10}\right) I\right) = \text{Null}\,(A + 0.1\,I)$

Now

$$[A + 0.1\,I \mid \underline{0}] = \left[\begin{array}{cc|c} 5/10 & 6/10 & 0 \\ 5/10 & 6/10 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 6/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So a basis for E_{λ_1} is

$$\left\{ \begin{bmatrix} -6/5 \\ 1 \end{bmatrix} \right\} \quad \text{or} \quad \left\{ \begin{bmatrix} -6 \\ 5 \end{bmatrix} \right\}$$

$\underline{E_{\lambda_2}} = \text{Null}\,(A - I)$.

Now

$$[A - I \mid \underline{0}] = \left[\begin{array}{cc|c} -6/10 & 6/10 & 0 \\ 5/10 & -5/10 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So a basis for E_{λ_2} is

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

(b) Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

where

$$a_{i1} + a_{i2} + \cdots + a_{in} = t \quad \text{for } 1 \le i \le n.$$

Consider $\underline{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{C}^n$. Then $\underline{x} \neq \underline{0}$

all entries = 1

and $A\underline{x} = \begin{bmatrix} a_{11}(1) + a_{12}(1) + \cdots + a_{1n}(1) \\ \vdots \\ a_{n1}(1) + a_{n2}(1) + \cdots + a_{nn}(1) \end{bmatrix}$

$$= \begin{bmatrix} t \\ \vdots \\ t \end{bmatrix} = t\underline{x}$$

By definition, $\underline{x}$ is an eigenvector of A with eigenvalue t.

9 (a) By definition, A is similar to a diagonal matrix D. Moreover, from a theorem in class, the eigenvalues of A are the diagonal entries of D. Now the determinants of similar matrices are equal. Hence,

$$\det(A) = \det(D) = \left(\begin{array}{c}\text{product of diagonal}\\ \text{entries of } D\end{array}\right)$$

$$= \left(\begin{array}{c}\text{product of eigenvalues}\\ \text{of } A\end{array}\right)$$

(b) With A and D as in part (b), we have $\operatorname{tr}(A) = \operatorname{tr}(D)$ since similar matrices have the same trace. Thus,

$$\operatorname{tr}(A) = \operatorname{tr}(D) = \left(\begin{array}{c}\text{sum of diagonal}\\ \text{entries of } D\end{array}\right)$$

$$= \left(\begin{array}{c}\text{sum of eigenvalues}\\ \text{of } A\end{array}\right)$$

10 (a) The characteristic polynomial of A is

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 4 & 3-\lambda & 2 \\ -3 & -2 & -3-\lambda \end{vmatrix}$$

$$\underset{(R_1 \to R_1+R_2+R_3)}{=} \begin{vmatrix} 2-\lambda & 2-\lambda & 2-\lambda \\ 4 & 3-\lambda & 2 \\ -3 & -2 & -3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 4 & 3-\lambda & 2 \\ -3 & -2 & -3-\lambda \end{vmatrix}$$

$$\underset{\substack{(R_2 \to R_2-4R_1) \\ (R_3 \to R_3+3R_1)}}{=} (2-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1-\lambda & -2 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} -1-\lambda & -2 \\ 1 & -\lambda \end{vmatrix}$$

$$= (2-\lambda)(\lambda^2+\lambda+2)$$

$$= -\lambda^3+\lambda^2+4$$

So, by the Cayley-Hamilton Theorem,

$$-A^3+A^2+4I = \underline{0}$$

Since 0 is not a root of the characteristic polynomial,

$0 \neq \det(A - 0I) = \det(A)$ and so A is invertible. Thus,

$$A^{-1}\left(-A^3 + A^2 + 4I\right) = A^{-1}\underline{0} = \underline{0}$$

and so

$$-A^2 + A + 4A^{-1} = \underline{0}$$

or

$$A^{-1} = \frac{1}{4}A^2 - \frac{1}{4}A$$

(b) The characteristic polynomial of B is

$$\det(B - \lambda I) = \begin{vmatrix} i-\lambda & -1 & 0 \\ 0 & -2-i-\lambda & 2 \\ 0 & -1-i & 1-\lambda \end{vmatrix}$$

$$= (i-\lambda)\begin{vmatrix} -2-i-\lambda & 2 \\ -1-i & 1-\lambda \end{vmatrix}$$

$$= (i-\lambda)\left[(-2-i-\lambda)(1-\lambda) + 2 + 2i\right]$$

$$= (i-\lambda)\left[-2-i-\lambda+2\lambda+i\lambda+\lambda^2+2+2i\right]$$

$$= (i-\lambda)\left[i + \lambda + i\lambda + \lambda^2\right]$$

$$= -\lambda^3 - \lambda^2 - \lambda - 1$$

So, by the Cayley-Hamilton Theorem,

$$-B^3 - B^2 - B - I = \underline{O}$$

Since 0 is not an eigenvalue of B we have

$$0 \neq \det(B - 0I) = \det(B)$$

and thus B is invertible. Hence,

$$B^{-1}(-B^3 - B^2 - B - I) = B^{-1}\underline{O} = \underline{O}$$

or

$$-B^2 - B - I - B^{-1} = \underline{O}$$

Thus, $B^{-1} = -B^2 - B - I$.