MATH 2090, Fall 2018 **Dr. S. Cooper**

Tutorial Worksheet #8

Monday, November 19

Name and Student Number: ______________________________

Work the following exercises. *Explain all of your reasoning.* Remember to use good notation and full sentences. All cited exercises are from the textbook *Linear Algebra* by Larry Smith.

1. Let $T : V \to W$ be a linear transformation, where V and W are both 3-dimensional vector spaces. Suppose A is the matrix representation of T with respect to some bases of V and W. Suppose further that $\det(A) = 0$. Show that T is not surjective or injective.

2. Find all the eigenvalues of
$$A = \begin{bmatrix} 2 & 4 & -5 \\ -1 & -5 & 7 \\ 0 & -2 & 3 \end{bmatrix}.$$

3. $\lambda = 2$ is an eigenvalue of the matrix
$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}.$$
Find a basis for the corresponding eigenspace E_λ.

4. Recall that the trace of a matrix A, denoted $\mathrm{tr}(A)$, is the sum of its diagonal entries.
   (a) Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$. Compute $\mathrm{tr}(AB)$ and $\mathrm{tr}(BA)$.
   (b) Prove that for any $n \times n$ matrices A and B, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
   (c) Prove that if A and B are $n \times n$ similar matrices, then $\mathrm{tr}(A) = \mathrm{tr}(B)$.

5. Prove that if A is similar to B and B is similar to C, then A is similar to C for $n \times n$ matrices A, B and C.

6. Suppose the matrix A is diagonalizable with diagonal matrix D (i.e., A is similar to D), and let the matrix B be similar to A. Prove that B is also diagonalizable with diagonal matrix D.

7. Explain why the following matrices are diagonalizable. Also, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.
   (a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

8. (a) Let
$$A = \begin{bmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}.$$
Find the eigenvalues of A. For each eigenvalue, find a basis for the corresponding eigenspace.

   (b) Suppose A is a square matrix, and suppose that the sum of the entries in each row is equal to some constant $t \in \mathbb{C}$. For example, in part (a) both the rows have a sum of 1, so $t = 1$. Prove that t is an eigenvalue of A.

9. Suppose A is a diagonalizable matrix.

   (a) Prove that the determinant of A is equal to the product of the eigenvalues of A.

   (b) Prove that the trace of A is equal to the sum of the eigenvalues of A.

   Note: These two statements are true even for matrices which aren't diagonalizable.

10. In this question we further investigate the Cayley–Hamilton Theorem introduced on Tutorial Worksheet #7. You may accept the theorem holds without proof. It says:

    Cayley-Hamilton Theorem: Suppose A is an $n \times n$ matrix with characteristic polynomial $c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$ for some $c_0, \ldots, c_n \in \mathbb{C}$. Then $c_nA^n + \cdots + c_1A + c_0I = \mathbf{0}$ where I is the $n \times n$ identity matrix, and $\mathbf{0}$ is the $n \times n$ zero matrix.

    (a) Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 4 & 3 & 2 \\ -3 & -2 & -3 \end{bmatrix}$. Show that $A^{-1} = \frac{1}{4}A^2 - \frac{1}{4}A$.

    (b) Let $B = \begin{bmatrix} i & -1 & 0 \\ 0 & -2-i & 2 \\ 0 & -1-i & 1 \end{bmatrix}$. Write B^{-1} as a polynomial in B. That is, write B^{-1} as $c_kB^k + \cdots + c_1B + c_0I$ for some $c_1, \ldots, c_k \in \mathbb{C}$ and some positive integer k.