Tutorial Worksheet #7
Some Sample Solutions

1. (a) $\det(A) = 3(1) - 2(2) = -1$

(b) $\det(A) = 4\det\left(\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\right) - 0\det\left(\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}\right)$

$$+3\det\left(\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\right)$$

$$= 4\big(2(-2)-(-1)(3)\big) + 3\big((1)(-1)-0\big)$$

$$= 4(-4+3) + 3(-1)$$

$$= 4(-1) - 3 = -7$$

(c) A is upper triangular and so

$\det(A)$ = product of diagonal entries of A

$$= (4)(2)(-1)(4) = -32$$

(d) $\det(A^T) = \det(A) = -32$

2. A is invertible if and only if $\det(A) \neq 0$

Now

$\det(A) = 2\begin{vmatrix} 1 & -1 \\ p & -2 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ 4 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 4 & p \end{vmatrix}$

$= 2(-2+p) - 3(-2+4) + 1(p-4)$

$= -4 + 2p - 6 + p - 4$

$= 3p - 14$

So A is invertible if and only if

$\det(A) = 3p - 14 \neq 0$

which is true if and only if

$$p \neq \frac{14}{3}$$

Hence, p can be any scalar such that $p \neq \frac{14}{3}$.

3 Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

Then

$$\det(tA) = \det\left(\begin{bmatrix} ta_{11} & \cdots & ta_{1n} \\ \vdots & & \\ ta_{n1} & \cdots & ta_{nn} \end{bmatrix}\right)$$

If $t=0$, then $\det(tA) = \det(0 \text{ matrix})$
$= 0$
$= 0^n \det(A)$

So assume $t \neq 0$. By our basic properties of deteminants,

$$\det\left(\begin{bmatrix} ta_{11} & \cdots & ta_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right) = t\det(A)$$

and so

$$\det\left(\begin{bmatrix} ta_{11} & \cdots & ta_{1n} \\ ta_{21} & \cdots & ta_{2n} \\ a_{31} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right) = t(t\det(A)) = t^2\det(A)$$

and so

$$\det\left(\begin{bmatrix} ta_{11} & \cdots & ta_{1n} \\ ta_{21} & \cdots & ta_{2n} \\ ta_{31} & \cdots & ta_{3n} \\ a_{41} & \cdots & a_{4n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right) = t(t^2\det(A)) = t^3\det(A)$$

Continuing in this fashion, we have

$$\det\left(\begin{bmatrix} ta_{11} & \cdots & ta_{1n} \\ \vdots & & \vdots \\ ta_{(n-1)1} & \cdots & ta_{(n-1)n} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right) = t^{n-1}\det(A)$$

and hence

$$\det(tA) = \det\left(\begin{bmatrix} ta_{11} & \cdots & ta_{1n} \\ \vdots & & \\ ta_{n1} & \cdots & ta_{nn} \end{bmatrix}\right)$$

$$= t\left(t^{n-1}\det(A)\right)$$

$$= t^{n}\det(A)$$

4. Let A be a 5×5 skew-symmetric matrix.

Then

$$\det(A) = \det(A^T) = \det(-A)$$

$$= (-1)^5 \det(A)$$

(by Exercise 3 with $t = -1$)

So $\det(A) = (-1)^5 \det(A) = -\det(A)$

But the only scalar which is equal to its negative is 0.

Thus, $\det(A) = 0$.

5. Let A be an $n \times n$ matrix such that $A^3 = I$

Then $\det(A^3) = \det(A)\det(A)\det(A)$
$= \det(I)$
$= 1$

Call $x = \det(A)$. Then we have shown
$$x^3 = 1$$

If $A \in M_{n\times n}(\mathbb{R})$ then we have $x = 1$ and so $\det(A) = 1 \neq 0$ showing A is invertible.

If $A \in M_{n\times n}(\mathbb{C})$, then we can use DeMoivre's Theorem to solve for x

Doing so yields

$$x = 1$$
$$x = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$x = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

So $\det(A) \in \left\{1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\right\}$

Again we see $\det(A) \neq 0$ and so A is invertible.

6) Let A and B be $n\times n$ invertible matrices. Then $\det(A) \neq 0$ and $\det(B) \neq 0$. Hence,

$$\det(AB) = (\det(A))(\det(B)) \neq 0$$

and

$$\det(BA) = (\det(B))(\det(A)) \neq 0$$

Therefore, both AB and BA are invertible.

7 Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Note that if $\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then

$$A\underline{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\underline{x}$$

Thus, by definition 2 is an eigenvalue of A.

Similarly, if $\underline{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then

$$A\underline{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\underline{x}$$

and again by definition, 0 is an eigenvalue of A.

8 Let $A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{n \times n}$

If $\underline{v} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{F}^n$ then observe that

$$A\underline{v} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} n \\ \vdots \\ n \end{bmatrix} = n\underline{v}$$

So, by definition, n is an eigenvalue of A.

Further, if $\underline{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{F}^n$, then

$$A\underline{w} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = 0\underline{w}$$

Again, by definition, 0 is an eigenvalue of A.

9) We have

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = [T]_{\mathcal{B}}^{\mathcal{B}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.1 & 0.8 \\ 0.5 & 0.4 & 0.1 \\ 0.15 & 0.2 & 0.65 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus, by definition, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of T with eigenvalue $\lambda = 1$.

10 (a) If $\operatorname{rank}(A) = n$, then
$$\dim(\operatorname{Im}(T)) = \operatorname{rank}(T) = n.$$
Thus, $\operatorname{Im}(T)$ is an n-dimensional subspace of W. Since $\dim(W) = n$ we must therefore have
$$W = \operatorname{Im}(T)$$ and so T is surjective.

Therefore, by the Rank-Nullity Theorem

$$n = \dim(V) = \operatorname{rank}(T) + \operatorname{Nullity}(T)$$
$$= n + \operatorname{Nullity}(T)$$

We conclude that
$$\operatorname{nullity}(T) = \dim(\ker(T)) = 0$$

so that $\ker(T) = \{\underline{0}_V\}$. Hence T is also injective.

Therefore, T is an isomorphism $\blacksquare$

(b) Suppose that $\lambda = 0$ is an eigenvalue of T. Then, by definition, there exists a non-zero vector $\underline{x} \in V$ such that

$$T(\underline{x}) = \lambda \underline{x} = 0\underline{x} = \underline{0}_W$$

Thus, $\underline{x} \in Ker(T)$ so that $Ker(T) \neq \{\underline{0}_V\}$

We conclude that

$$\dim(Ker(T)) = nullity(T) \geq 1.$$

So, by the Rank-Nullity Theorem,

$$n = \dim(V) = Rank(T) + Nullity(T)$$

and thus $Rank(T) \leq n-1$

Hence, $Im(T)$ is a subspace of W such that

$$\dim(Im(T)) = Rank(T) \leq n-1 < n = \dim(W)$$

That is, $Im(T) \neq W$ and so indeed

T is not surjective $\boxtimes$

11 (a) $\det(A-\lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}\right)$

$= (1-\lambda)(4-\lambda) - 6$

$= 4 - 5\lambda + \lambda^2 - 6$

$= \lambda^2 - 5\lambda - 2$

(b) $\det(B-\lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 0 & -1 \\ 2 & 2-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix}\right)$

$= (2-\lambda)\det\left(\begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}\right)$

$= (2-\lambda)\left[(1-\lambda)^2 + 1\right]$

$= (2-\lambda)\left[1 - 2\lambda + \lambda^2 + 1\right]$

$= (2-\lambda)(\lambda^2 - 2\lambda + 2)$

$= 2\lambda^2 - 4\lambda + 4 - \lambda^3 + 2\lambda^2 - 2\lambda$

$= -\lambda^3 + 4\lambda^2 - 6\lambda + 4$

c) For the matrix A:

$$A^2 - 5A - 2I = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

For the matrix B:

$$-B^3 + 4B^2 - 6B + 4I$$

$$= -\begin{bmatrix} -2 & 0 & -2 \\ 16 & 8 & -2 \\ 2 & 0 & -2 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 & -2 \\ 7 & 4 & 1 \\ 2 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 0 & -6 \\ 12 & 12 & 6 \\ 6 & 0 & 6 \end{bmatrix}$$

$$+ \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 2 \\ -16 & -8 & 2 \\ -2 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -8 \\ 28 & 16 & 4 \\ 8 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 0 & -6 \\ 12 & 12 & 6 \\ 6 & 0 & 6 \end{bmatrix}$$

$$+ \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$