MATH 2090, Fall 2018 **Dr. S. Cooper**

Tutorial Worksheet #7

Monday, November 5

Name and Student Number: ______________________________

Work the following exercises. *Explain all of your reasoning.* Remember to use good notation and full sentences. All cited exercises are from the textbook *Linear Algebra* by Larry Smith.

1. Compute the determinants of the following matrices:

   (a) $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 4 & 0 & 3 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 4 & 2 & 1 & -1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

   (d) The transpose of the matrix in part (c)

2. Determine the value(s) of p so that the matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & -1 \\ 4 & p & -2 \end{bmatrix}$$

   is invertible.

3. Let A be an $n \times n$ matrix. Prove that $\det(tA) = t^n \det(A)$ for all scalars t.

4. A square matrix A is **skew-symmetric** if $A^T = -A$. Show that if A is a 5×5 skew-symmetric matrix, then $\det(A) = 0$.

5. Suppose A is a square matrix such that $A^3 = I$, where I is the identity matrix. Prove that A is invertible.

6. Suppose A and B are invertible $n \times n$ matrices. Prove that AB and BA are invertible $n \times n$ matrices.

7. Show that 2 and 0 are eigenvalues of

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

8. Let $n \geq 2$ and let A be an $n \times n$ matrix with every entry equal to 1. Show that n and 0 are both eigenvalues of A.

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map such that

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 01 & 0.1 & 0.8 \\ 0.5 & 0.4 & 0.1 \\ 0.15 & 0.2 & 0.65 \end{bmatrix}$$

where $\mathcal{B}$ is the standard basis for $\mathbb{R}^3$. Prove that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of T.

10. Suppose $T : V \to W$ is a linear transformation between n-dimensional vector spaces, and let A be the matrix corresponding to T relative to bases $\mathcal{B}$ of V and $\mathcal{C}$ of W.

    (a) Prove that if $\text{rank}(A) = n$, then T is an isomorphism.

    (b) Prove that if 0 is an eigenvalue of T, then T is not surjective.

11. Compute the characteristic polynomial of the following matrices.

    (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

    (b) $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

    (c) For the characteristic polynomial obtained in part (a), replace all occurrences of λ with the matrix A, and replace any constants c by cI where I is the 2×2 identity matrix. What is the resulting 2×2 matrix? Repeat this for part (b) and the matrix B.

    If you are surprised, or at least intrigued, by your answer for part (c), you have just witnessed two instances of the Cayley-Hamilton Theorem!