Tutorial Worksheet #6
Some Sample Solutions

(1) Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be the standard bases of $\mathbb{R}^4, \mathbb{R}^7, \mathbb{R}^3$ respectively.

By a theorem from class, we know

$$[S \circ T]_{\mathcal{B}}^{\mathcal{D}} = [S]_{\mathcal{C}}^{\mathcal{D}} [T]_{\mathcal{B}}^{\mathcal{C}} = BA$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 4 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & 3 & 1 \\ 3 & 2 & -5 & 0 \\ 0 & 0 & 7 & 1 \\ 4 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 4 & 1 \\ 1 & 2 & -1 & 0 \end{bmatrix}$$

Thus, by definition of matrix representations,

$$(S \circ T)(1,-1,1,-1) = [S \circ T]_{\mathcal{B}}^{\mathcal{D}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}_{\mathcal{B}}$$

$$= \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 4 & 1 \\ 1 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$$

2) From a theorem in class, if we let
$\mathcal{B}$ = standard basis of $\mathbb{R}^3$,
$\mathcal{C}$ = standard basis of $\mathbb{R}^2$, then

$$[3S-7T]_{\mathcal{B}}^{\mathcal{C}} = 3[S]_{\mathcal{B}}^{\mathcal{C}} - 7[T]_{\mathcal{B}}^{\mathcal{C}}$$

$$= 3A - 7B$$

$$= \begin{bmatrix} 18 & -3 & 6 \\ 6 & 12 & 3 \end{bmatrix} - \begin{bmatrix} -35 & 0 & -49 \\ 14 & -7 & 63 \end{bmatrix}$$

$$= \begin{bmatrix} 53 & -3 & 55 \\ -8 & 19 & -60 \end{bmatrix}$$

Thus,

$$(3S-7T)(1,2,3) = [3S-7T]_{\mathcal{B}}^{\mathcal{C}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{B}}$$

$$= \begin{bmatrix} 53 & -3 & 55 \\ -8 & 19 & -60 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 212 \\ -150 \end{bmatrix}$$

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We first find $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$:

We have

- $1 = 1(1) + 0x + 0(3/2x^2 - 1/2) + 0(5/2x^3 - 3/2x)$

$\Rightarrow [1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

- $x = 0(1) + 1x + 0(3/2x^2 - 1/2) + 0(5/2x^3 - 3/2x)$

$\Rightarrow [x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

- $x^2 = 1/3(1) + 0x + 2/3(3/2x^2 - 1/2) + 0(5/2x^3 - 3/2x)$

$\Rightarrow [x^2]_{\mathcal{B}} = \begin{bmatrix} 1/3 \\ 0 \\ 2/3 \\ 0 \end{bmatrix}$

- $x^3 = 0(1) + 3/5x + 0(3/2x^2 - 1/2) + 2/5(5/2x^3 - 3/2x)$

$\Rightarrow [x^3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3/5 \\ 0 \\ 2/5 \end{bmatrix}$

Thus, $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \begin{bmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 2/5 \end{bmatrix}$

We can now easily change coordinates from $\mathcal{C}$ to $\mathcal{B}$.

(a) Let $p(x) = 1 + x + x^2 + x^3$
Then,

$$[p]_{\mathcal{B}} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} [p]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 2/5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4/3 \\ 8/5 \\ 2/3 \\ 2/5 \end{bmatrix}$$

(b) Let $p(x) = 2 - ix + (1+i)x^3$
Then,

$$[p]_{\mathcal{B}} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} [p]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 2/5 \end{bmatrix} \begin{bmatrix} 2 \\ -i \\ 0 \\ 1+i \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1/5\,(3-2i) \\ 0 \\ 2/5\,(1+i) \end{bmatrix}$$

(c) Let $p(x) = 3/2 + 5i/2\,x - 3/2\,x^2 - 5i/2\,x^3$

Then,

$$[p]_{\mathcal{B}} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P}\,[p]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 2/5 \end{bmatrix} \begin{bmatrix} 3/2 \\ 5i/2 \\ -3/2 \\ -5i/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}$$

(4) (a) We have by definition that

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \left[\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{B}} \right]$$

$$= \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

For $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ we recall that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\right)^{-1}$

Now $\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & -1/3 & -1/3 & 2/3 \\ 0 & 0 & 1 & -1/3 & 2/3 & -1/3 \end{array}\right]$$

And so,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\right)^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & -1/3 \end{bmatrix}$$

(b) You can find $[T]_{\mathcal{C}}^{\mathcal{C}}$ using the definition.

Another approach is to note that

$$[T]_{\mathcal{C}}^{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,[T]_{\mathcal{B}}^{\mathcal{B}}\,\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5 We let $\mathcal{B}$ = standard basis of $\mathbb{R}^3$

We also let $B = [T]_{\mathcal{C}}^{\mathcal{C}}$

We know that there exist invertible matrices P and Q such that

$$A = PBQ^{-1}$$

where

$$P = [I_{\mathbb{R}^3}]_{\mathcal{C}}^{\mathcal{B}} \quad \text{and} \quad Q = [I_{\mathbb{R}^3}]_{\mathcal{C}}^{\mathcal{B}}$$

with $I_{\mathbb{R}^3} : \mathbb{R}^3 \to \mathbb{R}^3$ the identity transformation on $\mathbb{R}^3$

Now $A = PBQ^{-1} \iff P^{-1}AQ = B$

Let's find P^{-1} and Q:

(i) $P^{-1} = [I_{\mathbb{R}^3}]_{\mathcal{B}}^{\mathcal{C}}$

Now $I_{\mathbb{R}^3}(1,0,0) = (1,0,0)$

$$= 0\underline{v_1} + 0\underline{v_2} + \underline{v_3}$$

$$\Rightarrow [I_{\mathbb{R}^3}(1,0,0)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$I_{\mathbb{R}^3}(0,1,0) = (0,1,0)$$
$$= 0\underline{v_1} + \underline{v_2} - \underline{v_3}$$

$$\Rightarrow [I_{\mathbb{R}^3}(0,1,0)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$I_{\mathbb{R}^3}(0,0,1) = (0,0,1) = \underline{v_1} - \underline{v_2} + 0\underline{v_3}$$

$$\Rightarrow [I_{\mathbb{R}^3}(0,0,1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Thus, $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

(ii) $Q = [I_{\mathbb{R}^3}]_{\mathcal{C}}^{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Therefore,

$$B = [T]_{\mathcal{C}}^{\mathcal{C}} = P^{-1}AQ$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ -2 & -2 & -2 \\ 3 & 2 & 2 \end{bmatrix}$$