Tutorial Worksheet #5
Some Sample Solutions

1. Let $\underline{x}, y \in \mathbb{F}^n$ and $\alpha \in \mathbb{F}$. We have (from basic properties of matrices):

(i) $T(\underline{x}+\underline{y}) = A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y}$

$= T(\underline{x}) + T(\underline{y})$

and

(ii) $T(\alpha \underline{x}) = A(\alpha \underline{x}) = \alpha (A\underline{x})$

$= \alpha T(\underline{x})$

Thus, T is a linear transformation.

2. $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\underline{e_1}}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\underline{e_2}}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\underline{e_3}} \right\}$

Now $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \dfrac{2(1+0+0)}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}$

$= 1/3 \, \underline{e_1} - 2/3 \, \underline{e_2} - 2/3 \, \underline{e_3}$

$\Rightarrow \left[T(\underline{e_1})\right]_{\mathcal{B}} = \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}$

$$T(\underline{e_2}) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{2(0+1+0)}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\0\end{bmatrix} - 2/3\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}-2/3\\1/3\\-2/3\end{bmatrix} = -2/3\,\underline{e_1} + 1/3\,\underline{e_2} - 2/3\,\underline{e_3}$$

$$\Rightarrow [T(\underline{e_2})]_{\mathcal{B}} = \begin{bmatrix}-2/3\\1/3\\-2/3\end{bmatrix}$$

Finally, $T(\underline{e_3}) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix} - \frac{2(0+0+1)}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$

$$= \begin{bmatrix}0\\0\\1\end{bmatrix} - 2/3\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}-2/3\\-2/3\\1/3\end{bmatrix}$$

$$\Rightarrow [T(\underline{e_3})]_{\mathcal{B}} = \begin{bmatrix}-2/3\\-2/3\\1/3\end{bmatrix}$$

So,

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \left[[T(\underline{e_1})]_{\mathcal{B}} \quad [T(\underline{e_2})]_{\mathcal{B}} \quad [T(\underline{e_3})]_{\mathcal{B}} \right]$$

$$= \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$

3) $\mathcal{B} = \{1, x, x^2\}$ = standard basis of $P_2(\mathbb{R})$

$\mathcal{C} = \{1, x, x^2, x^3\}$ = standard basis of $P_3(\mathbb{R})$

We want to find $[T]_{\mathcal{B}}^{\mathcal{C}}$

Now

$T(1) = x = 0(1) + 1(x) + 0x^2 + 0x^3$

$$\Rightarrow [T(1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$T(x) = x \cdot x = 0(1) + 0(x) + 1x^2 + 0x^3$

$$\Rightarrow [T(x)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(x^2) = x \cdot x^2 = 0(1) + 0(x) + 0(x^2) + 1(x^3)$$

$$\Rightarrow [T(x^2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $[T]_{\mathcal{B}}^{\mathcal{C}} = \left[[T(1)]_{\mathcal{C}} \quad [T(x)]_{\mathcal{C}} \quad [T(x^2)]_{\mathcal{C}} \right]$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4) By definition of matrix representation of a linear transformation, we have

$$[T(\underline{x})]_{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}} [\underline{x}]_{\mathcal{B}}$$

where $\mathcal{B}$ is the standard basis of $\mathbb{R}^4$ and $\mathcal{C}$ is the standard basis of $\mathbb{R}^7$

Now if $\underline{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, then $[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

and so

$$[T(\underline{x})]_{\mathcal{E}} = [T((1,2,3,4))]_{\mathcal{E}}$$

$$= \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 4 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & 3 & 1 \\ 3 & 2 & -5 & 0 \\ 0 & 0 & 7 & 1 \\ 4 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 14 \\ 8 \\ 11 \\ -8 \\ 25 \\ 7 \end{bmatrix}$$

Since $\mathcal{E}$ is the standard basis, we have

$T(1,2,3,4) = (2,14,8,11,-8,25,7)$

(5) $T: \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$T((x,y,z)) = (3y+4z, 3x, 4x)$$

(a) Let $\mathcal{B} = \{\underset{\underline{e_1}}{(1,0,0)}, \underset{\underline{e_2}}{(0,1,0)}, \underset{\underline{e_3}}{(0,0,1)}\}$

be the standard basis for $\mathbb{R}^3$

Now

$$T(\underline{e}_1) = T((1,0,0)) = (0,3,4) = 0\underline{e}_1 + 3\underline{e}_2 + 4\underline{e}_3$$

$$\therefore \quad [T(\underline{e}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$$

$$T(\underline{e}_2) = T((0,1,0)) = (3,0,0) = 3\underline{e}_1 + 0\underline{e}_2 + 0\underline{e}_3$$

$$\therefore \quad [T(\underline{e}_2)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$T(\underline{e}_3) = T((0,0,1)) = (4,0,0) = 4\underline{e}_1 + 0\underline{e}_2 + 0\underline{e}_3$$

$$\therefore \quad [T(\underline{e}_3)]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \Big[[T(\underline{e}_1)]_{\mathcal{B}} \quad [T(\underline{e}_2)]_{\mathcal{B}} \quad [T(\underline{e}_3)]_{\mathcal{B}} \Big]$$

$$= \begin{bmatrix} 0 & 3 & 4 \\ 3 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

(b) Let $\mathcal{B} = \{(0,4,-3), (5,3,4), (5,-3,-4)\}$, where $\underline{e}_1 = (0,4,-3)$, $\underline{e}_2 = (5,3,4)$, $\underline{e}_3 = (5,-3,-4)$.

We have

$$T(\underline{e}_1) = T((0,4,-3)) = (0,0,0) = 0\underline{e}_1 + 0\underline{e}_2 + 0\underline{e}_3$$

$$\Rightarrow [T(\underline{e}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T(\underline{e}_2) = T((5,3,4)) = (25,15,20) = 5(5,3,4)$$
$$= 0\underline{e}_1 + 5\underline{e}_2 + 0\underline{e}_3$$

$$\Rightarrow [T(\underline{e}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$T(\underline{e}_3) = T((5,-3,-4)) = (-25,15,20) = -5(5,-3,-4)$$

$$= 0\underline{e}_1 + 0\underline{e}_2 - 5\underline{e}_3$$

$$\Rightarrow [T(\underline{e}_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}$$

Thus,

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \Big[\, [T(\underline{e}_1)]_{\mathcal{B}} \quad [T(\underline{e}_2)]_{\mathcal{B}} \quad [T(\underline{e}_3)]_{\mathcal{B}}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

(6) $T: \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$T((x,y)) = (x+3y,\ 3x+y)$$

We have 2 bases for $\mathbb{R}^2$:

$$\mathcal{B} = \{(1,1),\ (1,-1)\}$$

$$\mathcal{C} = \{(1,0),\ (0,1)\}.$$

(a) We have

$$T((1,1)) = (4,4) = 4(1,1) + 0(1,-1)$$

$$\Rightarrow \left[T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$T((1,-1)) = (-2,2) = 0(1,1) - 2(1,-1)$$

$$\Rightarrow \left[T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\therefore [T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$$

(b) $T(1,1) = (4,4) = 4(1,0) + 4(0,1)$

$$\Rightarrow \left[T\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$T(1,-1) = (-2,2) = -2(1,0) + 2(0,1)$$

$$\Rightarrow \left[T\begin{pmatrix}1\\-1\end{pmatrix}\right]_{\mathcal{C}} = \begin{bmatrix}-2\\2\end{bmatrix}$$

$$\therefore [T]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix}4 & -2\\4 & 2\end{bmatrix}$$

7 Let $\mathcal{B} = \{\underline{v}_1, \dots, \underline{v}_n\}$ be a basis for V and $\mathcal{C} = \{\underline{w}_1, \dots, \underline{w}_m\}$ be a basis for W. Further, let $T: V \to W$ be a linear transformation. and $A = [T]_{\mathcal{B}}^{\mathcal{C}}$

$(\Rightarrow)$ Assume $\underline{v} \in \text{Ker}(T)$. Then $T(\underline{v}) = \underline{0}_W$

Now we want to show that

$$A[\underline{v}]_{\mathcal{B}} = \underline{0}_{\mathbb{F}^m}$$

We have that

$$A[\underline{v}]_{\mathcal{B}} = [T]_{\mathcal{B}}^{\mathcal{C}}[\underline{v}]_{\mathcal{B}} = [T(\underline{v})]_{\mathcal{C}}$$

$$= [\underline{0}_W]_{\mathcal{C}}$$

$$= \begin{bmatrix}0\\ \vdots \\0\end{bmatrix} \in \mathbb{F}^m$$

since $\underline{0}_W = 0\underline{w}_1 + \cdots + 0\underline{w}_m$.

($\Leftarrow$) Conversely, assume $[\underline{v}]_{\mathcal{B}} \in \text{Null}(A)$.

Then, by definition, $A[\underline{v}]_{\mathcal{B}} = \underline{0}_{\mathbb{F}^m}$

That is,

$$\underline{0}_{\mathbb{F}^m} = A[\underline{v}]_{\mathcal{B}} = [T]_{\mathcal{B}}^{\mathcal{C}} [\underline{v}]_{\mathcal{B}}$$
$$= [T(\underline{v})]_{\mathcal{C}}$$

So $T(\underline{v}) = 0\underline{w}_1 + \cdots + 0\underline{w}_m$

We conclude that $\underline{v} \in \text{Ker}(T)$ ☒

8 Let

(a)
$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
= standard basis for $M_{2\times 2}(\mathbb{R})$

$\mathcal{C} = \{1, x, x^2\}$ = standard basis for $P_2(\mathbb{R})$.

We have

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1 - 2x + 2x^2$$
$$\Rightarrow \left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = -3 + 6x - 6x^2$$

$$\Rightarrow \left[T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathcal{C}} = \begin{bmatrix} -3 \\ 6 \\ -6 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = 0 + 2x + 0x^2$$

$$\Rightarrow \left[T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\right]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 + 2x + 2x^2$$

$$\Rightarrow \left[T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\therefore [T]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 1 & -3 & 0 & 1 \\ -2 & 6 & 2 & 2 \\ 2 & -6 & 0 & 2 \end{bmatrix}$$

(b) To find a basis for $\operatorname{Ker}(T)$ and $\operatorname{Im}(T)$ we find bases for $\operatorname{Null}(A)$ and $\operatorname{Col}(A)$, where $A = [T]_{\mathcal{B}}^{\mathcal{C}}$

Now $[T]_{\mathcal{B}}^{\mathcal{C}} \rightsquigarrow \begin{bmatrix} 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So a basis for Col(A) is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Converting from our coordinate vectors, a basis for Im(T) is then

$$\{1 - 2x + 2x^2, 2x\}$$

Also, solving the system given by $A\underline{x} = \underline{0}$ yields

$$\begin{array}{cccc|c} & & & & \\ \end{array}$$

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

We have 2 free variables! Let $x_2 = s$, $x_4 = t$. Then

$$x_3 = -2x_4 = -2t$$

$$x_1 = 3x_2 - x_4 = 3s - t$$

So $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s-t \\ s \\ -2t \\ t \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

So $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for Null(A)

Converting from our coordinate vectors a basis for Ker(T) is then

$$\left\{ \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} \right\}$$