Tutorial Worksheet #4
Some Sample Solutions

1. We first show that $\{1+2x+x^2, 1+x^2, 1+x\}$ is linearly independent.

If $a,b,c \in \mathbb{R}$ such that
$$a(1+2x+x^2)+b(1+x^2)+c(1+x) = 0x^2+0x+0$$
then
$$\left.\begin{array}{r} a+b+c=0 \\ 2a+c=0 \\ a+b=0 \end{array}\right] \Rightarrow \quad a=-b \Rightarrow -b+b+c=0$$
$$\Rightarrow \quad c=0$$
Now $2a+c = 2a+0 = 0 \Rightarrow a=0$
$$\Rightarrow b=-a=0$$
Thus, we must have $a=b=c=0$ and so the set of polynomials is linearly independent.

Since, $\dim(P_2(\mathbb{R}))=3$, the 3 linearly independent polynomials form a basis for $P_2(\mathbb{R})$ by a theorem in class.

2. $\mathcal{U} = \{a+bx+cx^2 \in P_2(\mathbb{R}) \mid a+3b+9c=0\}$

$= \{a+bx+cx^2 \in P_2(\mathbb{R}) \mid a=-3b-9c\}$

$= \{(-3b-9c)+bx+cx^2 \in P_2(\mathbb{R})\}$

$= \{b(x-3)+c(x^2-9) \in P_2(\mathbb{R})\}$

$= \text{Span}(\{x-3, x^2-9\}) \subseteq P_2(\mathbb{R})$

<u>Claim</u> $\mathcal{B} = \{x-3, x^2-9\}$ is a basis for $\mathcal{U}$

(i) $\text{Span}(\mathcal{B}) = \mathcal{U}$ by above!

(ii) Let $a, b \in \mathbb{R}$ such that

$$a(x-3) + b(x^2-9) = \underline{0} = 0 + 0x + 0x^2$$

Then
$$\left.\begin{array}{r} -3a - 9b = 0 \\ a = 0 \\ b = 0 \end{array}\right] \Rightarrow a = b = 0$$

Thus, $\mathcal{B}$ is linearly independent

By (i) and (ii) $\mathcal{B}$ is a basis for $\mathcal{U}$.

(3) $\mathcal{U} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \,\middle|\, a + d = 0 \right\}$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \,\middle|\, d = -a \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \right\}$$

$$= \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$$

$$= \text{Span}\left(\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \right)$$

Let $\mathcal{B} = \left\{ \underline{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

Claim: $\mathcal{B}$ is a basis for $\mathcal{U}$

Pf (i) $\text{Span}(\mathcal{B}) = \mathcal{U}$ by above

(ii) Let $a, b, c \in \mathbb{R}$ such that

$$a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then $\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Leftrightarrow a = b = c = 0$$

$\therefore \mathcal{B}$ is linearly independent. $\boxtimes$

By definition, $\dim(\mathcal{U}) = 3$

(4) Let $a, b \in \mathbb{R}$ with

$$a(x + x^3) + b(1 + x^2) = 0 + 0x + 0x^2 + 0x^3$$

Then

$b = 0$
$a = 0$
$b = 0$
$a = 0$

and so the 2 given polynomials are linearly independent

However, $\dim(P_3(\mathbb{R})) = 4$ and so any basis for $P_3(\mathbb{R})$ has 4 elements. Thus, the 2 polynomials given cannot be a basis for $P_3(\mathbb{R})$

Finally, $\{x+x^3, 1+x^2\}$ Span the vector space $\mathrm{Span}(\{x+x^3, 1+x^2\})$ and is linearly independent. Thus, $\{x+x^3, 1+x^2\}$ is a basis for

$\mathrm{Span}(\{x+x^3, 1+x^2\})$ and so

$$\dim\left(\mathrm{Span}\left(\{x+x^3, 1+x^2\}\right)\right) = 2$$

5) Let V be a vector space with $\dim(V) = n$. Let $\{\underline{v_1}, \ldots, \underline{v_s}\} \subseteq V$ with $s > n$. Also, let $\{\underline{u_1}, \ldots, \underline{u_n}\}$ be a basis for V.
Assume, $\{\underline{v_1}, \ldots, \underline{v_s}\}$ is linearly independent. Since $V = \mathrm{Span}(\{\underline{u_1}, \ldots, \underline{u_n}\})$, a theorem from class says that $s \leq n$. This is a contradiction, and so our most recent assumption that $\{\underline{v_1}, \ldots, \underline{v_s}\}$ is linearly independent must be false.

Thus, $\{\underline{v_1}, \ldots, \underline{v_s}\}$ is linearly dependent.

6. Let V be an n-dimensional vector space with basis $\{\underline{v_1}, \ldots, \underline{v_n}\}$

Let $\{\underline{u_1}, \ldots, \underline{u_K}\} \subseteq V$ such that $K < n$.

Assume that $V = \text{Span}(\{\underline{u_1}, \ldots, \underline{u_K}\})$

By a theorem from class we can find a subset E of $\{\underline{u_1}, \ldots, \underline{u_K}\}$ such that E is linearly independent and

$$\text{Span}(E) = \text{Span}(\{\underline{u_1}, \ldots, \underline{u_K}\}) = V$$

But then, E is a basis for V and so $\dim(V)$ = the number of vectors in E $\leq K$

This contradicts our assumption that $\dim(V) = n > K$. Thus,

$$V \neq \text{Span}(\{\underline{u_1}, \ldots, \underline{u_K}\}) \quad \square$$

7. Suppose that $\alpha_1, \ldots, \alpha_K, \gamma$ are scalars such that

$$\alpha_1 \underline{v_1} + \cdots + \alpha_K \underline{v_K} + \gamma \underline{v_{K+1}} = \underline{0}$$

If $\gamma \neq 0$, then we have

$$\underline{v_{K+1}} = \left(-\frac{\alpha_1}{\gamma}\right)\underline{v_1} + \cdots + \left(-\frac{\alpha_K}{\gamma}\right)\underline{v_K}$$

so that $\underline{v}_{k+1} \in \text{Span}(\{\underline{v}_1, \ldots, \underline{v}_k\})$ which is a contradiction. Hence, $\gamma = 0$.

Thus, $\alpha_1 \underline{v}_1 + \cdots + \alpha_k \underline{v}_k + (0)\underline{v}_{k+1}$

$$= \alpha_1 \underline{v}_1 + \cdots + \alpha_k \underline{v}_k = \underline{0}$$

and so each $\alpha_j = 0$ since $\{\underline{v}_1, \ldots, \underline{v}_k\}$ is linearly independent.

Therefore we must have
$$\alpha_1 = \cdots = \alpha_k = \gamma = 0 \qquad \blacksquare$$

8 ($\Rightarrow$) Assume that $\{\underline{A}_1, \ldots, \underline{A}_n\}$ are linearly independent. Then by a theorem in class, $\{\underline{A}_1, \ldots, \underline{A}_n\}$ is a basis for V (since $\dim V = n$). Hence,

$$V = \text{Span}(\{\underline{A}_1, \ldots, \underline{A}_n\}) \quad \text{and}$$

$$n = \dim(V) = \dim(\text{Span}(\{\underline{A}_1, \ldots, \underline{A}_n\}))$$

($\Leftarrow$) Conversely, assume that

$$\dim(\text{Span}(\{\underline{A}_1, \ldots, \underline{A}_n\})) = n.$$

Now $\text{Span}(\{\underline{A}_1, \ldots, \underline{A}_n\})$ is a subspace of V. So, by a Corollary from class, $V = \text{Span}(\{\underline{A}_1, \ldots, \underline{A}_n\})$

By a theorem from class, it follows that $\{\underline{A_1}, \ldots, \underline{A_n}\}$ is a basis for V and thus is linearly independent. $\boxtimes$

9. $S \cap T$ is a subspace of V by a previous worksheet. Since V is finite-dimensional, so is $S \cap T$.
Let $A = \{\underline{x_1}, \ldots, \underline{x_k}\}$ be a basis for $S \cap T$.
By the Basis Extension Theorem we have bases

$$\{\underline{x_1}, \ldots, \underline{x_k}, \underline{y_1}, \ldots, \underline{y_m}\} \text{ for } S$$

and

$$\{\underline{x_1}, \ldots, \underline{x_k}, \underline{z_1}, \ldots, \underline{z_n}\} \text{ for } T.$$

Now $S+T = \mathrm{Span}(\{\underline{x_1}, \ldots, \underline{x_k}, \underline{y_1}, \ldots, \underline{y_m}, \underline{z_1}, \ldots, \underline{z_n}\})$

Claim $\{\underline{x_1}, \ldots, \underline{x_k}, \underline{y_1}, \ldots, \underline{y_m}, \underline{z_1}, \ldots, \underline{z_n}\}$ are linearly independent.

Pf Let $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n$ be scalars such that

$$(\star) \quad \underline{0} = \alpha_1 \underline{x_1} + \cdots + \alpha_k \underline{x_k} + \beta_1 \underline{y_1} + \cdots + \beta_m \underline{y_m} + \gamma_1 \underline{z_1} + \cdots + \gamma_n \underline{z_n}$$

Then

$$-\gamma_1 \underline{z_1} - \cdots - \gamma_n \underline{z_n} = \alpha_1 \underline{x_1} + \cdots + \alpha_k \underline{x_k} + \beta_1 \underline{y_1} + \cdots + \beta_m \underline{y_m}$$

$$\in \mathrm{Span}\{\underline{x_1}, \ldots, \underline{x_k}, \underline{y_1}, \ldots, \underline{y_m}\} = S$$

Since $-\gamma_1 \underline{z_1} - \cdots - \gamma_n \underline{z_n} \in T$, we have
$-\gamma_1 \underline{z_1} - \cdots - \gamma_n \underline{z_n} \in S \cap T$

Thus, $-\gamma_1 \underline{z_1} - \cdots - \gamma_n \underline{z_n} = a_1 \underline{x_1} + \cdots + a_k \underline{x_k}$

for some scalars $a_1, \ldots, a_k$

So, $a_1 \underline{x_1} + \cdots + a_k \underline{x_k} + \gamma_1 \underline{z_1} + \cdots + \gamma_n \underline{z_n} = \underline{0}$

and thus $a_1 = \cdots = a_k = \gamma_1 = \cdots = \gamma_n = 0$

(since $\{\underline{x_1}, \ldots, \underline{x_k}, \underline{z_1}, \ldots, \underline{z_n}\}$ is a basis hence linearly independent)

Therefore, returning to ($\star$) gives

$$\underline{0} = \alpha_1 \underline{x_1} + \cdots + \alpha_k \underline{x_k} + \beta_1 \underline{y_1} + \cdots + \beta_m \underline{y_m}$$

$$\Rightarrow \alpha_1 = \cdots = \alpha_k = \beta_1 = \cdots = \beta_m = 0$$

(since $\{\underline{x_1}, \ldots, \underline{x_k}, \underline{y_1}, \ldots, \underline{y_m}\}$ is a basis) $\square$

Thus, $\{\underline{x_1}, \ldots, \underline{x_k}, \underline{y_1}, \ldots, \underline{y_m}, \underline{z_1}, \ldots, \underline{z_n}\}$ is a basis for $S+T$

We conclude that

$$\begin{aligned} \dim(S) + \dim(T) &= (k+m) + (k+n) \\ &= k + (m+k+n) \\ &= \dim(S \cap T) + \dim(S+T) \end{aligned}$$ $\boxtimes$

10 By definition

$$q(x) = -1(1+x^2) + 1(x-3x^2) + 2(1+x-3x^2)$$

$$= -1 - x^2 + x - 3x^2 + 2 + 2x - 6x^2$$

$$= 1 + 3x - 10x^2$$

11 Let $\mathcal{B} = \{\underline{v_1}, \ldots, \underline{v_n}\}$.
Let $x, y \in V$ and $t \in \mathbb{F}$.
Since $\mathcal{B}$ is a basis for V, we have scalars $a_1, \ldots, a_n$ and $b_1, \ldots, b_n \in \mathbb{F}$ such that

$$\underline{x} = a_1 \underline{v_1} + \cdots + a_n \underline{v_n}$$

and

$$\underline{y} = b_1 \underline{v_1} + \cdots + b_n \underline{v_n}$$

Then $\underline{x} + \underline{y} = (a_1+b_1)\underline{v_1} + \cdots + (a_n+b_n)\underline{v_n}$

Thus,

$$[\underline{x}+\underline{y}]_{\mathcal{B}} = \begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$= [\underline{x}]_{\mathcal{B}} + [\underline{y}]_{\mathcal{B}}$$

Also,

$$t\underline{x} = ta_1\underline{v}_1 + \cdots + ta_n v_n$$

Thus,

$$t[\underline{x}]_B = t\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ta_1 \\ \vdots \\ ta_n \end{bmatrix} = [t\underline{x}]_B$$

12 (a) $\underline{v} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\therefore [\underline{v}]_S = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Now $\underline{v} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = a\begin{pmatrix} 2 \\ -1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\Leftrightarrow \begin{aligned} 2 &= 2a - b \\ 5 &= -a + b \end{aligned}$$

$$\Leftrightarrow a = 7, b = 12$$

$$\therefore [\underline{v}]_B = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

(b) $\underline{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\therefore \ [\underline{v}]_S = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Also, $\underline{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 2 \\ -1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\Leftrightarrow \quad \begin{aligned} 1 &= 2a - b \\ 0 &= -a + b \end{aligned}$$

$$\Leftrightarrow \quad a = 1 = b$$

$$\therefore \ [\underline{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(13) Let $p(x), q(x) \in P_n(\mathbb{R})$ and $\gamma \in \mathbb{R}$. Then

(i) $$\begin{aligned} ev_a((p+q)(x)) &= (p+q)(a) \\ &= p(a) + q(a) \\ &= ev_a(p(x)) + ev_a(q(x)) \end{aligned}$$

(ii) $$\begin{aligned} ev_a((\gamma p)(x)) &= (\gamma p)(a) \\ &= \gamma p(a) \\ &= \gamma \cdot ev_a(p(x)) \end{aligned}$$

14 Let $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} s \\ t \end{pmatrix} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$.

Then

(i) $T([x,y] + [s,t]) = T([x+s, y+t])$

$$= [x+s+y+t, y+t, x+s]$$

$$= [x+y, y, x] + [s+t, t, s]$$

$$= T([x,y]) + T([s,t])$$

(ii) $T(\alpha[x,y]) = T([\alpha x, \alpha y])$

$$= [\alpha x + \alpha y, \alpha y, \alpha x]$$

$$= \alpha[x+y, y, x]$$

$$= \alpha T([x,y])$$

15 (a) Consider $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$. Then

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\neq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

(b) By a result in class, if T were linear we would have $T(\underline{0}) = \underline{0}$

But $T\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix}$

$\therefore$ T is <u>not</u> linear.

(16) Since T is linear, we have

$$T([2,1]) = T([1,1] + [1,0])$$
$$= T([1,1]) + T([1,0])$$
$$= 3 + 4$$
$$= 7$$

(17) $T \circ S([1,1,1])$

$$= T(S([1,1,1]))$$
$$= T(2,2,2,3)$$
$$= (4,5)$$