MATH 2090, Fall 2018 **Dr. S. Cooper**

Tutorial Worksheet #4

Monday, October 1

Name and Student Number: ______________________________

Work the following exercises. *Explain all of your reasoning.* Remember to use good notation and full sentences. All cited exercises are from the textbook *Linear Algebra* by Larry Smith.

1. Prove that $\{1+2x+x^2, 1+x^2, 1+x\}$ is a basis for $\mathcal{P}_2(\mathbb{R})$.
2. Find a basis (and verify that it is!) for the subspace $\mathcal{U} = \{p(x) \in \mathcal{P}_2(\mathbb{R}) \mid p(3) = 0\}$.
3. Let $\mathcal{U}$ be the subspace of $M_{2\times 2}(\mathbb{R})$ given by
$$\mathcal{U} := \{A \in M_{2\times 2}(\mathbb{R}) : \operatorname{tr}(A) = 0\}$$
where $\operatorname{tr}(A)$ is the trace of the matrix A (defined in an example from class). Find a basis for $\mathcal{U}$ and prove it is a basis. What is the dimension of $\mathcal{U}$?
4. (Section 7.2 #4) Are the polynomials $\{x+x^3, 1+x^2\}$ a set of linearly independent vectors in $\mathcal{P}_3(\mathbb{R})$? If so, is $\{x+x^3, 1+x^2\}$ a basis for $\mathcal{P}_3(\mathbb{R})$? What is the dimension of $Span(\{x+x^3, 1+x^2\})$?
5. Show that any finite subset of $> n$ vectors in an n-dimensional vector space are linearly dependent.
6. Prove that a set of fewer than n vectors in an n-dimensional vector space V cannot span V.
7. Suppose $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ is a linearly independent set in a vector space V. Suppose $\mathbf{v_{k+1}}$ is not in the span of $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$. Prove that $\{\mathbf{v_1}, \ldots, \mathbf{v_k}, \mathbf{v_{k+1}}\}$ is linearly independent.
8. (Section 6.4 #17) Let $\mathbf{A_1}, \ldots, \mathbf{A_n}$ be vectors in a vector space V. Suppose that $n = \dim(V)$. Show that $\{\mathbf{A_1}, \ldots, \mathbf{A_n}\}$ are linearly independent if and only if $Span(\{\mathbf{A_1}, \ldots, \mathbf{A_n}\})$ has dimension n.
9. (Section 6.4 #10) Suppose that $\mathcal{S}$ and $\mathcal{T}$ are subspaces of a finite-dimensional vector space V. Show that
$$\dim(\mathcal{S}+\mathcal{T}) = \dim(\mathcal{S}) + \dim(\mathcal{T}) - \dim(\mathcal{S}\cap\mathcal{T}).$$
[*Hint:* Choose a basis for $\mathcal{S}\cap\mathcal{T}$, extend it to a basis for $\mathcal{S}$, and extend it to a basis for $\mathcal{T}$. Show that the result spans V and count the number of vectors that occur twice.]
10. $\mathcal{B} = \{1+x^2, x-3x^2, 1+x-3x^2\}$ is a basis for $\mathcal{P}_2(\mathbb{R})$. Find the polynomial $q(x) \in \mathcal{P}_2(\mathbb{R})$ such that
$$[q(x)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

11. Let V be a finite-dimensional vector space over the field $\mathbb{F}$ with basis $\mathcal{B}$. Prove that

$$[\mathbf{x}]_{\mathcal{B}} + [\mathbf{y}]_{\mathcal{B}} = [\mathbf{x} + \mathbf{y}]_{\mathcal{B}}$$

and

$$t[\mathbf{x}]_{\mathcal{B}} = [t\mathbf{x}]_{\mathcal{B}}$$

for all $\mathbf{x}, \mathbf{y} \in V$ and $t \in \mathbb{F}$.

12. Let $\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis for $\mathbb{R}^2$, and let $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ be another basis for $\mathbb{R}^2$.

    (a) Let $\mathbf{v} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. Compute $[\mathbf{v}]_{\mathcal{S}}$ and $[\mathbf{v}]_{\mathcal{B}}$.

    (b) Let $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Compute $[\mathbf{v}]_{\mathcal{S}}$ and $[\mathbf{v}]_{\mathcal{B}}$.

13. For any $a \in \mathbb{R}$, define the evaluation map

$$\mathrm{ev}_a : \mathcal{P}_n(\mathbb{R}) \to \mathbb{R}$$

by $\mathrm{ev}_a(p) := p(a)$. Prove that ev_a is a linear transformation.

14. (Section 8.7 #1) Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(x, y) = (x + y, y, x)$$

is a linear transformation.

15. (Section 8.7 #2) Show that the following functions are not linear transformations:

    (a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (xy, y, x)$

    (b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x + y + z, 1)$

16. (Section 8.7 #9) Let $T : \mathbb{R}^2 \to \mathbb{R}$ be a linear transformation. Suppose that $T([1, 1]) = 3$ and $T([1, 0]) = 4$. Calculate $T([2, 1])$.

17. (Section 8.7 #14) Let $S : \mathbb{R}^3 \to \mathbb{R}^4$, $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformations defined by

$$S([a_1, a_2, a_3]) = [a_1 + a_2, a_1 + a_3, a_2 + a_3, a_1 + a_2 + a_3]$$

and

$$T([b_1, b_2, b_3, b_3]) = [b_1 + b_2, b_3 + b_4].$$

Compute the composition $T \circ S([1, 1, 1])$.