Tutorial Worksheet #3
Some Sample Solutions

1 (a) Let $a, b, c \in \mathbb{R}$ such that

$$a(1+x+x^2) + b(5-2x+2x^2) + c(-2+3x+x^2) = 0 + 0x + 0x^2$$

After combining like terms and equating coefficients we must have

$$\begin{aligned} a + 5b - 2c &= 0 \\ a - 2b + 3c &= 0 \\ a + 2b + c &= 0 \end{aligned}$$

To solve this system of equations we set up our augmented matrix and row-reduce (details left for the student):

$$\left[\begin{array}{ccc|c} 1 & 5 & -2 & 0 \\ 1 & -2 & 3 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So the only solution to our system is $a = b = c = 0$.
By $\text{def}^{\underline{n}}$, our given polynomials are linearly independent.

(b) Let $a, b \in \mathbb{R}$ such that

$$a\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $\begin{bmatrix} a+2b & a+2b & 0 \\ a & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Thus, $a=0$ and $a+2b=0$

We conclude that $a=0$ and $b=0$
By def$^{\underline{n}}$, the 2 given matrices are linearly independent.

(c) Let $a, b \in \mathbb{C}$ such that

$$a\begin{bmatrix} i \\ 2i \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then $ai + 2b = 0$ and $a2i + b = 0$

Say $a = x+iy$, $b = s+ti$ with $x, y, s, t \in \mathbb{R}$
Thus,

$$0 = (x+iy)i + 2b = xi - y + (2s + 2ti) = (2s-y) + (x+2t)i$$

and

$$0 = (x+iy)2i + (s+ti) = 2xi - 2y + s + ti$$
$$= (2x+t)i + (s-2y)$$

So
$$\left.\begin{array}{r} x+2t=0 \\ 2s-y=0 \\ 2x+t=0 \\ s-2y=0 \end{array}\right] \Rightarrow \begin{array}{l} s=2y \\ 2s=y \\ t=-2x \\ x=-2t \end{array} \Rightarrow \begin{array}{l} 4y=y \\ 4x=x \end{array} \Rightarrow \begin{array}{c} y=0 \\ x=0 \\ \Downarrow \\ s=0=t \end{array}$$

Thus $a = b = 0$

2 (a) $4\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}4\\4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

and $4 \neq 0$ and $1 \neq 0$

(b) Suppose $a, b, c, d \in \mathbb{R}$ such that

$$a(1+3x+x^2) + b(-1+x^2) + c(5+x) + d(3x^2) = 0 + 0x + 0x^2$$

Then equating coefficients gives

$$\begin{aligned} a - b + 5c &= 0 \\ 3a + c &= 0 \\ a + b + 3d &= 0 \end{aligned}$$

We augment the system and row-reduce

$$\left[\begin{array}{cccc|c} 1 & -1 & 5 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 5 & 0 & 0 \\ 0 & 1 & -14/3 & 0 & 0 \\ 0 & 0 & 1 & 9/13 & 0 \end{array}\right]$$

So $c = -9/13\, d$

$b = 14/3\, c$

$a = b - 5c$

d is a free variable

Let $d = 13$ so that $c = -9$, $b = -42$, $a = 3$

Thus,

$$3(1+3x+x^2) - 42(-1+x^2) - 9(5+x) + 13(3x^2) = 0$$

Since the scalars a,b,c,d are not all 0, the given polynomials are linearly dependent.

ⓒ Let $a, b, c, d \in \mathbb{R}$ such that

$$a\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 5 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$\begin{bmatrix} a-b+5c & 3a+c \\ 0 & a+b+3d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So that
$$a-b+5c=0$$
$$3a+c=0$$
$$a+b+3d=0$$

We solved this system in part b!
One solution is $a=3$, $b=-42$, $c=-9$, $d=13$

Since not all the scalars are 0, the given matrices are linearly dependent.

(d) Let $a, b \in \mathbb{C}$ such that

$$a\begin{bmatrix} 1+i \\ i \end{bmatrix} + b\begin{bmatrix} 2i \\ -1+i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then

$$a + ai + 2bi = 0$$
$$ai - b + bi = 0$$

That is,

$$a + (a+2b)i = 0$$
$$-b + (a+b)i = 0$$

Let $a = x+iy$, $b = s+it$ where $x, y, s, t \in \mathbb{R}$

Then

$$0 = (x+iy) + (x+iy+2s+2it)i = x+iy+xi-y+2si-2t$$
$$= (x-y-2t) + (y+x+2s)i$$

and

$$0 = -(s+it) + (x+iy+s+it)i = -s-it+xi-y+si-t$$
$$= (-s-y-t) + (-t+x+s)i$$

Thus,
$$x-y-2t=0$$
$$x+y+2s=0$$
$$-y-s-t=0$$
$$x+s-t=0$$

We augment the system and solve:

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 1 & 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$y = -s-t$ $\qquad$ s, t are free variables

$x = y+2t$

Let $s = t = 1$ so that $b = 1 + i$
and $y = -1 - 1 = -2 \Rightarrow x = -2 + 2 = 0$
$\therefore a = -2i$

<u>Check</u>:

$$-2i\begin{bmatrix}1+i\\ i\end{bmatrix} + (1+i)\begin{bmatrix}2i\\ -1+i\end{bmatrix}$$

$$= \begin{bmatrix}-2i+2\\ 2\end{bmatrix} + \begin{bmatrix}2i-2\\ -2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Since a and b are not both 0, the vectors are linearly dependent.

(e) Let $\mathcal{B} = \{\underline{v}, \underline{v}, \underline{u_1}, \ldots, \underline{u_K}\} \subseteq V$ where V is a vector space over the field $\mathbb{F}$.
Then we have

$$1\underline{v} - 1\underline{v} + 0\underline{u_1} + \cdots + 0\underline{u_K} = \underline{0}$$

Since not all the scalars are 0, $\mathcal{B}$ is linearly dependent.

3 Let $a, b \in \mathbb{R}$ such that

$$a\sin x + b\cos x = 0.$$

Let $x = 0$. Then $a\sin(0) + b\cos(0) = 0$ becomes $a(0) + b(1) = 0$. That is, $b = 0$.
Now let $x = \pi/2$ to have

$$a\sin(\pi/2) + b\cos(\pi/2) = 0$$

Equivalently,

$$a(1) + b(0) = 0$$

and so $a = 0$.

Since $a = 0$ and $b = 0$, $\sin x$ and $\cos x$ are linearly independent in V.

4 Let $\underline{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\underline{v_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\underline{v_3} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\underline{v_4} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Note that $\underline{v_1} + \underline{v_2} - \underline{v_3} - \underline{v_4} = \underline{0}$

So that E is linearly dependent.
Remove $\underline{v_4}$ and let $F = \{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$. Then if $a, b, c \in \mathbb{R}$ such that

$$a\underline{v_1} + b\underline{v_2} + c\underline{v_3} = \underline{0} \iff \begin{bmatrix} a+b \\ a+b+c \\ c \\ b+c \\ a+b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{aligned} a &= -b \\ a &= -b-c \\ c &= 0 \\ b &= -c \\ a &= -b-c \end{aligned}$$

which is true if and only if

$$a=b=c=0$$

Thus, F is linearly independent.

<u>Claim</u> : $\text{Span}(F) = \mathcal{U} = \text{Span}(E)$.

Let $y \in \text{Span}(F)$. Then there are scalars in $\mathbb{R}$ such that

$$\begin{aligned} y &= a_1 \underline{v_1} + a_2 \underline{v_2} + a_3 \underline{v_3} \\ &= a_1 \underline{v_1} + a_2 \underline{v_2} + a_3 \underline{v_3} + 0 \underline{v_4} \\ &\in \text{Span}\{\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}\} = \text{Span}(E). \end{aligned}$$

Conversely, let $\underline{x} \in \text{Span}(E)$. Then there are scalars in $\mathbb{R}$ such that

$$\begin{aligned} \underline{x} &= a_1 \underline{v_1} + a_2 \underline{v_2} + a_3 \underline{v_3} + a_4 \underline{v_4} \\ &= a_1 \underline{v_1} + a_2 \underline{v_2} + a_3 \underline{v_3} + a_4(\underline{v_1} + \underline{v_2} - \underline{v_3}) \\ &= (a_1 + a_4)\underline{v_1} + (a_2 + a_4)\underline{v_2} + (a_3 - a_4)\underline{v_3} \\ &\in \text{Span}(\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}) = \text{Span}(F). \end{aligned}$$

5 (a) $\mathcal{U} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a+b=d \right\}$

$= \left\{ \begin{bmatrix} a & b \\ c & a+b \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$

$= \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$

$= \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{A_3} \right\}$

Suppose $a,b,c \in \mathbb{R}$ such that

$$a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This is true iff

$$\begin{bmatrix} a & b \\ c & a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and so we must have $a=b=c=0$

We conclude that $\{A_1, A_2, A_3\}$ is a linearly independent set of vectors.

By def$^{\underline{n}}$, a basis for $\mathcal{U}$ is

$\{A_1, A_2, A_3\}$.

(b) $\dim(\mathcal{U}) = 3$ by part (a)

6 Let $E = \{\underline{u_1}, \dots, \underline{u_K}\}$
and
$F = \{\underline{u_1}, \dots, \underline{u_K}, \underline{u_{K+1}}, \dots, \underline{u_m}\}$

Now E linearly dependent means that there are scalars $t_1, \dots, t_K$ <u>not all zero</u> such that

$$t_1 \underline{u_1} + \dots + t_K \underline{u_K} = \underline{0}$$

But then

$$t_1 \underline{u_1} + \dots + t_K \underline{u_K} + 0 \underline{u_{K+1}} + \dots + 0 \underline{u_K} = \underline{0}$$

Since $t_1, \dots, t_K$ are not all zero this means that F is linearly dependent by def$^{\underline{n}}$.

7 Let $E = \{\underline{u_1}, \dots, \underline{u_K}\}$
and
$F = \{\underline{u_1}, \dots, \underline{u_K}, \underline{u_{K+1}}, \dots, \underline{u_m}\}$

Assume that E is linearly dependent. Then, by #6, F is linearly dependent which contradicts our assumption that F is linearly independent. Hence, E must be linearly independent.

* This is called a proof by contrapositive *

8 By definition,

$$E' \cap E'' = \{\underline{x} \in V \mid \underline{x} \in E' \text{ and } \underline{x} \in E''\} \subseteq E'$$

Since E' is linearly independent, so is $E' \cap E''$ by #7

9 Note that

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathcal{U}$$

if and only if

$$0 = p(0) = a_0 + a_1(0) + \cdots + a_n(0) = a_0$$

So

$$\mathcal{U} = \{p(x) = a_1 x + \cdots + a_n x^n : a_i \in \mathbb{R}\}$$

$$= \operatorname{Span}(\{x, x^2, \ldots, x^n\})$$

Also if $t_1, \ldots, t_n \in \mathbb{R}$ such that $t_1 x + t_2 x^2 + \cdots + t_n x^n = 0x + \cdots + 0x^n$

Then, by equating coefficients, we must have

$t_1 = 0$
$t_2 = 0$
$\vdots$
$t_n = 0$

That is, $\{x, x^2, \ldots, x^n\}$ is also linearly independent.

Therefore, $\{x, x^2, \ldots, x^n\}$ is a basis for $\mathcal{U}$ and hence $\dim(\mathcal{U}) = n$