Tutorial Worksheet #2
Some Sample Solutions

1 (a) We want to show that $(-1)\underline{a} = -\underline{a}$ for all $\underline{a}$ in V. We have

$$\underline{a} + (-1)\underline{a} = 1\underline{a} + (-1)\underline{a} \quad \text{[multiplicative identity]}$$
$$= [1 + (-1)]\underline{a} \quad \text{[distributivity]}$$
$$= 0\underline{a}$$
$$= \underline{0} \quad \text{[Proposition in class]}$$

By Exercise 2 on this worksheet, it must be true that $(-1)\underline{a}$ is the additive inverse of $\underline{a}$. That is, $(-1)\underline{a} = -\underline{a}$. ☐

(b) We want to show that $r\underline{0} = \underline{0}$ for all $r \in \mathbb{F}$. We have

$$r\underline{0} = r(\underline{0} + \underline{0}) \quad \text{[additive identity]}$$
$$= r\underline{0} + r\underline{0} \quad \text{[distributivity]}$$

Now add $-r\underline{0}$ to both sides:

$$r\underline{0} + (-r\underline{0}) = [r\underline{0} + r\underline{0}] + (-r\underline{0})$$
$$r\underline{0} + (-r\underline{0}) = r\underline{0} + [r\underline{0} + (-r\underline{0})] \quad \text{(associativity)}$$
$$\underline{0} = r\underline{0} + \underline{0} \quad \text{[additive inverse]}$$
$$\underline{0} = r\underline{0} \quad \text{[additive identity]}$$

☐

2 Suppose $\underline{w} \in V$ satisfies the property

$$\underline{u} + \underline{w} = \underline{0}$$

Adding $-\underline{u}$ to both sides gives

$$(-\underline{u}) + [\underline{u} + \underline{w}] = (-\underline{u}) + \underline{0}$$

$$[(-\underline{u}) + \underline{u}] + \underline{w} = (-\underline{u}) + \underline{0} \quad \text{[associativity]}$$

$$\underline{0} + \underline{w} = (-\underline{u}) + \underline{0} \quad \text{[additive inverse]}$$

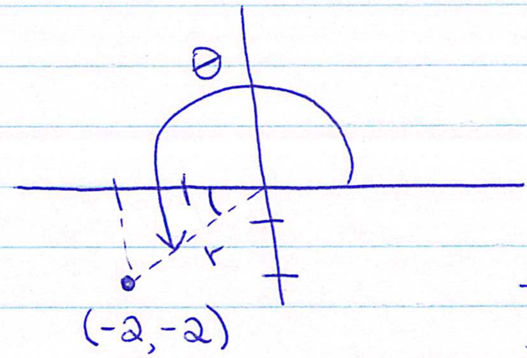
$$\underline{w} = -\underline{u} \quad \text{[additive identity]}$$

3 a) $(3+4i) + (22i) = 3 + 26i$

b) $(2+7i)(3i) = 6i + 21i^2 = 6i - 21$
$= -21 + 6i$

c) $|7+2i| = \sqrt{7^2 + 2^2} = \sqrt{49+4} = \sqrt{53}$

d)



θ = argument of $-2-2i$

$-2 = r\cos\theta$
$-2 = r\sin\theta$

Pythagorean Theorem $\Rightarrow (-2)^2 + (-2)^2 = r^2$

$$4 + 4 = r^2$$
$$\sqrt{8} = r$$
$$2\sqrt{2} = r$$

Thus, $\cos\theta = \dfrac{-2}{2\sqrt{2}} = \dfrac{-1}{\sqrt{2}}$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{-3\pi}{4}$$

We always express the argument as an angle between 0 and 2π. Thus the argument is

$$\frac{-3\pi}{4} + 2\pi = \frac{5\pi}{4}$$

(4) Let $z = a+bi$, $w = c+di \in \mathbb{C}$

(a) $|zw| = |(a+bi)(c+di)|$

$$= |(ac-bd) + (ad+bc)i|$$

$$= \sqrt{(ac-bd)^2 + (ad+bc)^2}$$

$$= \sqrt{a^2c^2 - 2acbd + b^2d^2 + a^2d^2 + 2adbc + b^2c^2}$$

$$= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}$$

$$= \sqrt{(a^2+b^2)(c^2+d^2)}$$

$$= \left(\sqrt{a^2+b^2}\right)\left(\sqrt{c^2+d^2}\right)$$

$$= |z|\,|w|$$

(b) $z\bar{z} = (a+bi)(a-bi)$

$= a^2 - abi + abi - b^2 i^2$

$= a^2 - b^2(-1)$

$= a^2 + b^2$

$= \left(\sqrt{a^2+b^2}\right)^2$

$= |z|^2$

(c) $\overline{zw} = \overline{(a+bi)(c+di)}$

$= \overline{(ac-bd) + (ad+bc)i}$

$= (ac-bd) - (ad+bc)i$

and

$\bar{z}\,\bar{w} = (a-bi)(c-di)$

$= ac - adi - bci + bdi^2$

$= (ac-bd) - (ad+bc)i$

5. Let $\mathcal{U}$ be a subspace of V such that $\mathcal{B} = \{\underline{v}_1, \ldots, \underline{v}_k\} \subseteq \mathcal{U}$.

To show that $\text{Span}(\mathcal{B}) \subseteq \mathcal{U}$ we need to show that any vector in $\text{Span}(\mathcal{B})$ is also in $\mathcal{U}$. So let $\underline{x} \in \text{Span}(\mathcal{B})$. This means we can write $\underline{x}$ as a linear combination of $\{\underline{v}_1, \ldots, \underline{v}_k\}$. That is,

$$\underline{x} = \alpha_1 \underline{v}_1 + \cdots + \alpha_k \underline{v}_k \quad \text{for some scalars}$$

$$\alpha_1, \ldots, \alpha_k \in \mathbb{F}.$$

Since each $\underline{v}_j \in \mathcal{U}$, we have $\alpha_j \underline{v}_j \in \mathcal{U}$ by closure under scalar multiplication. (since $\mathcal{U}$ is itself a vector space!) Thus, by closure under vector addition in $\mathcal{U}$, $\underline{x}$ is also in $\mathcal{U}$.

Therefore, $\text{Span}(\mathcal{B}) \subseteq \mathcal{U}$. $\boxtimes$

6. (a) $\mathcal{U}$ is a subspace of $\mathbb{R}^3$.
We proof this using the subspace test:
(i) $(0,0,0) \in \mathcal{U}$ since $0+0=0$
Thus $\mathcal{U}$ is non-empty.
(ii) Let (a_1, a_2, a_3) and $(b_1, b_2, b_3) \in \mathcal{U}$.
Then
$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1+b_1, a_2+b_2, a_3+b_3)$$

which is in $\mathcal{U}$ since

$$(a_1+b_1)+(a_2+b_2) = (a_1+a_2)+(b_1+b_2)$$
$$= 0+0 = 0$$

(iii) Let $\alpha \in \mathbb{R}$ and $(a_1, a_2, a_3) \in \mathcal{U}$. Then

$$\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3)$$

which is in $\mathcal{U}$ since $\alpha a_1 + \alpha a_2 = \alpha(a_1+a_2)$
$= \alpha(0)$
$= 0$

(b) $\mathcal{U} = \{(x_1, x_2, x_3) \mid x_1 + x_2 \geq 0\}$ is <u>not</u> a subspace of $\mathbb{R}^3$.
For example, let $\alpha = -1$ and $\underline{x} = (1, 1, 0)$. Now $1+1 = 2 \geq 0$ and so

$\underline{x} \in \mathcal{U}$. However,

$$\alpha \underline{x} = -1\underline{x} = -1(1,1,0) = (-1,-1,0) \notin \mathcal{U}$$

since $-1+(-1) = -2 < 0$

(7) Note that
$\underline{0} = 0 + 0x + 0x^2 \in \mathcal{U}$ since $0+0=0$
Thus $\mathcal{U}$ is non-empty.
Now let $p(x) = a_1 + b_1 x + c_1 x^2$ and
$q(x) = a_2 + b_2 x + c_2 x^2$
both be in $\mathcal{U}$. Then

$p(x)+q(x) = (a_1+a_2)+(b_1+b_2)x+(c_1+c_2)x^2$
is in $\mathcal{U}$ since

$$(b_1+b_2)+(c_1+c_2) = (b_1+c_1)+(b_2+c_2)$$
$$= a_1+a_2$$

Also, if $p(x)=a+bx+cx^2 \in \mathcal{U}$ and $\alpha \in \mathbb{R}$ then
$\alpha p(x) = \alpha a+\alpha bx+\alpha cx^2$ is in $\mathcal{U}$ since

$$\alpha b+\alpha c = \alpha(b+c) = \alpha a$$

So, by the Subspace Test, $\mathcal{U}$ is a subspace of $P_2(\mathbb{R})$.

8. $\mathcal{U} = \{A \in M_{2\times 2}(\mathbb{R}) : \det(A)=0\}$ is <u>not</u> a subspace of $M_{2\times 2}(\mathbb{R})$ since it fails to be closed under addition.

For example,

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

are both in $\mathcal{U}$ but

$A+B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ is not in $\mathcal{U}$ since

$$\det(A+B) = 3(2)-5(1) = 1 \neq 0$$

9 (a) $\mathcal{U} = \{ p(x) \in P_5(\mathbb{R}) \mid p(0) = 1 \}$ is <u>not</u> a subspace of $P_5(\mathbb{R})$. For example,

$p(x) = 1$ and $q(x) = x + 1$ are both in $\mathcal{U}$

but $p(x) + q(x) = x + 2$ is not in $\mathcal{U}$

since

$$p(0) + q(0) = 2 \neq 1$$

(b) $\mathcal{U} = \{ p(x) \in P_5(\mathbb{R}) : p(-x) = -p(x) \}$ is a subspace of $P_5(\mathbb{R})$

(i) Note that $\underline{0} = 0 + 0x + 0x^2 + 0x^3 + 0x^4 + 0x^5$ is in $\mathcal{U}$.

(ii) Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

$$q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5$$

both be in $\mathcal{U}$. Then

$$(p+q)(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 + (a_4 + b_4)x^4 + (a_5 + b_5)x^5$$

and

$$\begin{aligned}(p+q)(-x) &= (a_0 + b_0) + (a_1 + b_1)(-x) + (a_2 + b_2)(-x)^2 + (a_3 + b_3)(-x)^3 + (a_4 + b_4)(-x)^4 + (a_5 + b_5)(-x)^5 \\ &= a_0 - a_1 x + a_2(-x)^2 + a_3(-x)^3 + a_4(-x)^4 + a_5(-x)^5 \\ &\quad + b_0 - b_1 x + b_2(-x)^2 + b_3(-x)^3 + b_4(-x)^4 + b_5(-x)^5\end{aligned}$$

$$= p(-x) + q(-x) = -p(x) - q(x) = -(p(x)+q(x))$$

Showing that $p+q \in \mathcal{U}$.

iii Also, if $\alpha \in \mathbb{R}$ and $p(x) \in \mathcal{U}$, then with
$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$
we have

$$\alpha p(-x) = \alpha a_0 + \alpha a_1(-x) + \alpha a_2 (-x)^2 + \alpha a_3 (-x)^3 + \alpha a_4 (-x)^4 + \alpha a_5 (-x)^5$$

$$= \alpha(-p(x))$$

$$= -(\alpha p(x))$$

ie $\alpha p \in \mathcal{U}$

So by the Subspace Test, $\mathcal{U}$ is a subspace of $P_5(\mathbb{R})$

10 $\mathcal{U}$ is not a subspace of $\mathbb{C}^2$ since it fails to be closed under scalar multiplication with scalars from $\mathbb{C}$.

For example $\begin{pmatrix} i \\ -i \end{pmatrix} \in \mathcal{U}$

since $\mathrm{Re}(i) = \mathrm{Re}(-i) = 0$

but

$$i\begin{pmatrix} i \\ -i \end{pmatrix} = \begin{pmatrix} i^2 \\ -i^2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \notin \mathcal{U}$$

since $\mathrm{Re}(-1) = -1 \neq 1 = \mathrm{Re}(1)$

11 $\mathcal{U} = \{f : \mathbb{R} \to \mathbb{R} \mid f(3) + f(5) = 0\}$ is a subspace.

Certainly the zero function is in $\mathcal{U}$.
Let $f, g \in \mathcal{U}$ and $\alpha \in \mathbb{R}$. Then

$$
\begin{aligned}
(f+g)(3) + (f+g)(5) &= f(3) + g(3) + f(5) + g(5) \\
&= [f(3) + f(5)] + [g(3) + g(5)] \\
&= 0 + 0 = 0
\end{aligned}
$$

so that $f+g \in \mathcal{U}$
and

$$
\begin{aligned}
(\alpha f)(3) + (\alpha f)(5) &= \alpha f(3) + \alpha f(5) \\
&= \alpha (f(3) + f(5)) \\
&= \alpha (0) \\
&= 0
\end{aligned}
$$

so that $\alpha f \in \mathcal{U}$.

12 $\mathrm{Span}\{1+x, 1-x\}$

$$
\begin{aligned}
&= \{\alpha(1+x) + \beta(1-x) \mid \alpha, \beta \in \mathbb{R}\} \\
&= \{\alpha + \alpha x + \beta - \beta x \mid \alpha, \beta \in \mathbb{R}\} \\
&= \{(\alpha+\beta) + (\alpha-\beta)x \mid \alpha, \beta \in \mathbb{R}\}
\end{aligned}
$$

Writing

13 $\begin{bmatrix}-4\\-2\\2\\-6\end{bmatrix} = x_1\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + x_2\begin{bmatrix}2\\0\\0\\2\end{bmatrix} + x_3\begin{bmatrix}0\\2\\-1\\1\end{bmatrix}$

for some $x_1, x_2, x_3 \in \mathbb{R}$
is equivalent to solving the linear system of equations

$$-4 = x_1 + 2x_2$$
$$-2 = x_1 + 2x_3$$
$$2 = -x_3$$
$$-6 = x_1 + 2x_2 + x_3$$

Now $x_3 = -2 \Rightarrow -2 = x_1 + 2(-2)$

$$2 = x_1$$

and so

$$-4 = 2 + 2x_2$$
$$-6 = 2x_2$$
$$-3 = x_2$$

Check:

$$\begin{bmatrix}-4\\-2\\2\\-6\end{bmatrix} = 2\begin{bmatrix}1\\1\\0\\1\end{bmatrix} - 3\begin{bmatrix}2\\0\\0\\2\end{bmatrix} - 2\begin{bmatrix}0\\2\\-1\\1\end{bmatrix}$$

14 Writing

$$42x + 2x^2 - 6x^3 = a(1 + x + x^3) + b(2 + x^3) + c(2x - x^2 + x^3)$$

for some $a, b, c \in \mathbb{R}$ leads to a system of equations by comparing the coefficients of like terms:

$$a + 2b = 0$$
$$a + 2c = 42$$
$$-c = 2$$
$$a + b + c = -6$$

$c = -2 \Rightarrow a + 2c = 42$
$$a + 2(-2) = 42$$
$$a - 4 = 42$$
$$a = 46$$
$\therefore \; a + 2b = 0$
$$46 + 2b = 0$$
$$2b = -46$$
$$b = -23$$

But $46 - 23 - 2 \neq -6$

So there is no solution,

$42x + 2x^2 - 6x^3$ is not in Span($\mathcal{B}$)

15 (a) See Proposition 4.1.5 of your textbook

(b) Yes, $\mathcal{E}$ is a subspace of $\mathbb{R}^2$:

$\underline{0} = (0,0) = (2 \cdot 0, 0) \in \mathcal{E}$

If $\underline{x} = (2a, a)$, $\underline{y} = (2b, b) \in \mathcal{E}$ and $r \in \mathbb{R}$ then

$\underline{x} + \underline{y} = (2a + 2b, a+b) = (2(a+b), a+b) \in \mathcal{E}$

and

$r\underline{x} = r(2a, a) = (2ar, ar) \in \mathcal{E}$

(c) Yes, $\mathcal{B}$ is a subspace of $\mathbb{R}^2$:

$\underline{0} = (0,0) \in \mathcal{B}$

If $\underline{x} = (a,a)$, $\underline{y} = (b,b) \in \mathcal{B}$ and $r \in \mathbb{R}$ then

$\underline{x} + \underline{y} = (a+b, a+b) \in \mathcal{B}$ and

$r\underline{x} = (ra, ra) \in \mathcal{B}$

(d) $\mathcal{E} \cap \mathcal{B} = \{\underline{x} \in \mathbb{R}^2 \mid \underline{x} \in \mathcal{E} \text{ and } \underline{x} \in \mathcal{B}\}$

$= \{(a,b) \mid a = 2b \text{ and } a = b\}$

$= \{(a,b) \in \mathbb{R}^2 \mid b = 0 = a\}$

$= \{(0,0) \in \mathbb{R}^2\}$

(e) <u>No!</u> $\mathcal{E} \cup \mathcal{B}$ is <u>not</u> a subspace of $\mathbb{R}^2$

For example,

$(2,1) \in \mathcal{E}$ and so $(2,1) \in \mathcal{E} \cup \mathcal{B}$

and

$(1,1) \in \mathcal{B}$ and so $(1,1) \in \mathcal{E} \cup \mathcal{B}$

but

$(2,1) + (1,1) = (3,2)$ which is not in $\mathcal{E} \cup \mathcal{B}$

(f) $\mathcal{E} + \mathcal{B} = \{(2a, a) + (b,b) \mid a, b \in \mathbb{R}\}$

$= \{(2a+b, a+b) \mid a, b \in \mathbb{R}\}$

(16) Suppose $\underline{A} + \underline{B} = \underline{A}' + \underline{B}'$ for some $\underline{A}, \underline{A}' \in S$ and $\underline{B}, \underline{B}' \in T$. Then $\underline{A} - \underline{A}' = \underline{B} - \underline{B}'$. But the vector $\underline{x} = \underline{A} - \underline{A}' \in S$ and the vector $\underline{x} = \underline{B} - \underline{B}' \in T$ (by closure). That is this vector $x \in S \cap T = \{\underline{0}\}$.

So, $\underline{x} = \underline{A} - \underline{A}' = \underline{B} - \underline{B}' = \underline{0}$

We conclude that

$$\underline{A} = \underline{A}' \quad \text{and} \quad \underline{B} = \underline{B}'$$

<u>Example</u> In $V = \mathbb{R}^3$

Let $S = \{(x,y,0) \mid x,y \in \mathbb{R}\}$

$T = \{(0,w,z) \mid w,z \in \mathbb{R}\}$

Both S and T are subspaces of $\mathbb{R}^3$

Also, $S + T = \mathbb{R}^3$

Now $S \cap T \neq \{\underline{0}\}$ since

$(0,1,0) \in S$ and $(0,1,0) \in T$

Note that

$$\left.\begin{aligned} (1,1,1) &= \underbrace{(1,1,0)}_{\in S} + \underbrace{(0,0,1)}_{\in T} \\ &= \underbrace{(1,0,0)}_{\in S} + \underbrace{(0,1,1)}_{\in T} \end{aligned}\right]$$

2 different sums give $(1,1,1)$