MATH 2090, Fall 2018 **Dr. S. Cooper**

Tutorial Worksheet #2

Monday, September 17

Name and Student Number: ____

Work the following exercises. *Explain all of your reasoning.* Remember to use good notation and full sentences. All cited exercises are from the textbook *Linear Algebra* by Larry Smith.

1. Let V be a vector space over the field $\mathbb{F}$. Prove the following:
   (a) $(-1)\mathbf{a} = -\mathbf{a}$ for all $\mathbf{a} \in V$;
   (b) $r\mathbf{0} = \mathbf{0}$ for all $r \in \mathbb{F}$.
2. Prove that in any vector space V, the additive inverses are indeed unique. That is, for any vector $\mathbf{u} \in V$, show that $-\mathbf{u}$ is the only vector in V such $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
3. In the complex numbers, compute the following:
   (a) $(3 + 4i) + (22i)$
   (b) $(2 + 7i)(3i)$
   (c) $|7 + 2i|$
   (d) the argument of $-2 - 2i$
4. Let z and w be complex numbers. Using the definitions, prove the following:
   (a) $|zw| = |z| \cdot |w|$
   (b) $z\overline{z} = |z|^2$
   (c) $\overline{zw} = \overline{z} \cdot \overline{w}$
5. Let $\mathcal{B} = \{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ be a subset of a vector space V. In class we showed that $Span(\mathcal{B})$ is a subspace of V. Prove that $Span(\mathcal{B})$ is the smallest vector space containing $\mathcal{B}$. That is, show that if any subspace $\mathcal{U}$ of V contains $\mathcal{B}$, then $Span(\mathcal{B}) \subseteq \mathcal{U}$.
6. (Section 4.3 #1) Which of the following collections of vectors in $\mathbb{R}^3$ are linear subspaces?
   (a) $\mathcal{U} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 = 0\}$
   (b) $\mathcal{U} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 \geq 0\}$
7. Prove that $\mathcal{U} = \{a + bx + cx^2 : b + c = a\}$ is a subspace of $\mathcal{P}_2(\mathbb{R})$.
8. Is $\{A \in M_{2\times 2}(\mathbb{R}) : \det(A) = 0\}$ a subspace of $M_{2\times 2}(\mathbb{R})$?
9. Determine, with proof, whether the following subsets of $\mathcal{P}_5(\mathbb{R})$ are subspaces.
   (a) $\{p(x) \in \mathcal{P}_5(\mathbb{R}) : p(0) = 1\}$

(b) $\{p(x) \in \mathcal{P}_5(\mathbb{R}) : p(-x) = -p(x)\}$

10. Is

$$\mathcal{U} = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{C}^2 : Re(z_1) = Re(z_2) = 0 \right\}$$

a subspace of $\mathbb{C}^2$? Justify your answer.

11. Determine (with proof) if the set of functions $\{f \in Fun(\mathbb{R}) : f(3) + f(5) = 0\}$ is a subspace of the vector space $Fun(\mathbb{R})$.

12. (Section 4.3 #7) What is the span of $\{1 + x, 1 - x\}$ in $\mathcal{P}_1(\mathbb{R})$?

13. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^4.$$

Either express the vector

$$\begin{pmatrix} -4 \\ -2 \\ 2 \\ -6 \end{pmatrix}$$

as a linear combination of the vectors in $\mathcal{B}$ or show that it is not a vector in $Span(\mathcal{B})$.

14. Let $\mathcal{B} = \{1 + x + x^3, 2 + x^3, 2x - x^2 + x^3\} \subseteq \mathcal{P}_3(\mathbb{R})$. Either express $42x + 2x^2 - 6x^3$ as a linear combination of the vectors in $\mathcal{B}$ or show that it is not a vector in $Span(\mathcal{B})$.

15. (Section 4.3 # 27) Recall that if $\mathcal{U}$ and $\mathcal{W}$ are subspaces of a vector space V then we define the intersection of $\mathcal{U}$ and $\mathcal{W}$ as

$$\mathcal{U} \cap \mathcal{W} = \{\mathbf{x} \in V : \mathbf{x} \in \mathcal{U} \text{ and } \mathbf{x} \in \mathcal{W}\}$$

and the union of $\mathcal{U}$ and $\mathcal{W}$ as

$$\mathcal{U} \cup \mathcal{W} = \{\mathbf{x} \in V : \mathbf{x} \in \mathcal{U} \text{ or } \mathbf{x} \in \mathcal{W}\}.$$

We also define the sum

$$\mathcal{U} + \mathcal{W} = \{\mathbf{x} \in V \mid \mathbf{x} = \mathbf{u} + \mathbf{w} \text{ for some } \mathbf{u} \in \mathcal{U} \text{ and } \mathbf{w} \in \mathcal{W}\}.$$

(a) Prove that $\mathcal{U} \cap \mathcal{W}$ is always a subspace of V.

(b) Let $\mathcal{E} = \{(2a, a) \mid a \in \mathbb{R}\}$. Is $\mathcal{E}$ a subspace of $\mathbb{R}^2$?

(c) Let $\mathcal{B} = \{(b, b) \mid b \in \mathbb{R}\}$. Is $\mathcal{B}$ a subspace of $\mathbb{R}^2$?

(d) What is $\mathcal{E} \cap \mathcal{B}$?

(e) Is $\mathcal{E} \cup \mathcal{B}$ a subspace of $\mathbb{R}^2$?

(f) What is $\mathcal{E} + \mathcal{B}$?

16. (Section 4.3 # 11) Suppose that $\mathcal{S}$ and $\mathcal{T}$ are subspaces of the vector space V and that $\mathcal{S} \cap \mathcal{T} = \{\mathbf{0}\}$. Sow that every vector in $\mathcal{S} + \mathcal{T}$ can be written *uniquely* in the form $\mathbf{A} + \mathbf{B}$ with $\mathbf{A} \in \mathcal{S}$ and $\mathbf{B} \in \mathcal{T}$. Construct an example to show that this is false if $\mathcal{S} \cap \mathcal{T} \neq \{\mathbf{0}\}$.