Tutorial Worksheet #1
Some Sample Solutions

Q1 We first verify that we have closure under addition and scalar multiplication.

- Let $\underline{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $\underline{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \in M_{2\times3}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then

(i) $\underline{A} + \underline{B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix} \in M_{2\times3}(\mathbb{R})$

ie $M_{2\times3}(\mathbb{R})$ is closed under addition

(ii) $\lambda \underline{A} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \end{bmatrix} \in M_{2\times3}(\mathbb{R})$

ie $M_{2\times3}(\mathbb{R})$ is closed under scalar mult. by real scalars.

We next verify the 8 axioms from the def$^{\underline{n}}$ of a vector space.

① Commutativity:

$$\underline{A} + \underline{B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} & b_{13}+a_{13} \\ b_{21}+a_{21} & b_{22}+a_{22} & b_{23}+a_{23} \end{bmatrix} = \underline{B} + \underline{A}$$

② Associativity Let $\underline{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} \in M_{2\times3}(\mathbb{R})$

Then

$$(\underline{A}+\underline{B})+\underline{C} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} & a_{13}+b_{13}+c_{13} \\ a_{21}+b_{21}+c_{21} & a_{22}+b_{22}+c_{22} & a_{23}+b_{23}+c_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) & a_{13}+(b_{13}+c_{13}) \\ a_{21}+(b_{21}+c_{21}) & a_{22}+(b_{22}+c_{22}) & a_{23}+(b_{23}+c_{23}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} & b_{13}+c_{13} \\ b_{21}+c_{21} & b_{22}+c_{22} & b_{23}+c_{23} \end{bmatrix}$$

$$= \underline{A} + (\underline{B}+\underline{C})$$

3 $\underline{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the unique zero vector in $M_{2\times3}(\mathbb{R})$ such that

$$\underline{0}+\underline{A} = \underline{A}+\underline{0} = \underline{A}$$

Indeed

$$\underline{0}+\underline{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} 0+a_{11} & 0+a_{12} & 0+a_{13} \\ 0+a_{21} & 0+a_{22} & 0+a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Similarly, $\underline{A}+\underline{0} = \underline{A}$

So $M_{2\times3}(\mathbb{R})$ has an additive identity.

(4) Given $\underline{A}$, let $-\underline{A} = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$

Note that $-\underline{A} \in M_{2\times 3}(\mathbb{R})$ and

$$\underline{A} + (-\underline{A}) = \begin{bmatrix} a_{11}-a_{11} & a_{12}-a_{12} & a_{13}-a_{13} \\ a_{21}-a_{21} & a_{22}-a_{22} & a_{23}-a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underline{0}$$

So $\underline{A}$ has an additive inverse.

(5) <u>Multiplicative Identity</u>

$$1\underline{A} = \begin{bmatrix} 1a_{11} & 1a_{12} & 1a_{13} \\ 1a_{21} & 1a_{22} & 1a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$= \underline{A}$$

(6) <u>Distributivity</u> Let $r, s \in \mathbb{R}$. Then

$$(r+s)\underline{A} = \begin{bmatrix} (r+s)a_{11} & (r+s)a_{12} & (r+s)a_{13} \\ (r+s)a_{21} & (r+s)a_{22} & (r+s)a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} ra_{11}+sa_{11} & ra_{12}+sa_{12} & ra_{13}+sa_{13} \\ ra_{21}+sa_{21} & ra_{22}+sa_{22} & ra_{23}+sa_{23} \end{bmatrix}$$

$$= \begin{bmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \end{bmatrix} + \begin{bmatrix} sa_{11} & sa_{12} & sa_{13} \\ sa_{21} & sa_{22} & sa_{23} \end{bmatrix}$$

$$= r\underline{A} + s\underline{A}$$

7 **Distributivity** Let $r \in \mathbb{R}$. Then

$$r(\underline{A}+\underline{B}) = r\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} r(a_{11}+b_{11}) & r(a_{12}+b_{12}) & r(a_{13}+b_{13}) \\ r(a_{21}+b_{21}) & r(a_{22}+b_{22}) & r(a_{23}+b_{23}) \end{bmatrix}$$

$$= \begin{bmatrix} ra_{11}+rb_{11} & ra_{12}+rb_{12} & ra_{13}+rb_{13} \\ ra_{21}+rb_{21} & ra_{22}+rb_{22} & ra_{23}+rb_{23} \end{bmatrix}$$

$$= \begin{bmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \end{bmatrix} + \begin{bmatrix} rb_{11} & rb_{12} & rb_{13} \\ rb_{21} & rb_{22} & rb_{23} \end{bmatrix}$$

$$= r\underline{A} + r\underline{B}$$

8 Let $r, s \in \mathbb{R}$. Then

$$(rs)\underline{A} = \begin{bmatrix} rsa_{11} & rsa_{12} & rsa_{13} \\ rsa_{21} & rsa_{21} & rsa_{23} \end{bmatrix}$$

$$= \begin{bmatrix} r(sa_{11}) & r(sa_{12}) & r(sa_{13}) \\ r(sa_{21}) & r(sa_{21}) & r(sa_{23}) \end{bmatrix}$$

$$= r\begin{bmatrix} sa_{11} & sa_{12} & sa_{13} \\ sa_{21} & sa_{21} & sa_{23} \end{bmatrix}$$

$$= r(s\underline{A})$$

Q2 V is <u>not</u> a vector space since it does not have a multiplicative identity. For example, consider $(2,3) \in V$. Then

$$1 \cdot (2,3) = (1\cdot 2, 0) = (2,0) \neq (2,3)$$

Q3 Yes, $V \oplus W$ is a vector space over $\mathbb{F}$.

We first check closure under addition and scalar multiplication:

- Let $(\underline{A},\underline{B})$ and $(\underline{C},\underline{D}) \in V \oplus W$ and $\lambda \in \mathbb{F}$.

Then

(i) $(\underline{A},\underline{B}) + (\underline{C},\underline{D}) = (\underline{A}+\underline{C}, \underline{B}+\underline{D})$

Now $\underline{A}, \underline{C} \in V$ and V is closed under addition. Thus, $\underline{A}+\underline{C} \in V$. Similarly, $\underline{B}, \underline{D} \in W$ and W is closed under addition so that $\underline{B}+\underline{D} \in W$.

Hence, $(\underline{A}+\underline{C}, \underline{B}+\underline{D}) \in V \oplus W$ as desired.

(ii) $\lambda(\underline{A},\underline{B}) = (\lambda\underline{A}, \lambda\underline{B})$.

Since $\underline{A} \in V$, $\underline{B} \in W$ and both V and W are closed under scalar multiplication from $\mathbb{F}$, both $\lambda\underline{A} \in V$ and $\lambda\underline{B} \in W$. Thus,

$(\lambda\underline{A}, \lambda\underline{B}) \in V \oplus W$ as desired.

We next verify the 8 axioms from the def$^{\underline{n}}$ of a vector space.

Many properties are inherited from V & W.

① Commutativity:

commutativity in V commutativity in W

$$(\underline{A},\underline{B}) + (\underline{C},\underline{D}) = (\underline{A}+\underline{C},\ \underline{B}+\underline{D}) = (\underline{C}+\underline{A},\ \underline{D}+\underline{B})$$

$$= (\underline{C},\underline{D}) + (\underline{A},\underline{B})$$

② Associativity Let $(\underline{E},\underline{F}) \in V \oplus W$. Then

$$\big((\underline{A},\underline{B}) + (\underline{C},\underline{D})\big) + (\underline{E},\underline{F}) = (\underline{A}+\underline{C},\ \underline{B}+\underline{D}) + (\underline{E},\underline{F})$$

$$= \big((\underline{A}+\underline{C})+\underline{E},\ (\underline{B}+\underline{D})+\underline{F}\big)$$

$$= \big(\underline{A}+(\underline{C}+\underline{E}),\ \underline{B}+(\underline{D}+\underline{F})\big)$$

$$= (\underline{A},\underline{B}) + (\underline{C}+\underline{E},\ \underline{D}+\underline{F})$$

$$= (\underline{A},\underline{B}) + \big((\underline{C},\underline{D}) + (\underline{E},\underline{F})\big)$$

③ Additive Identity Since V and W are vector spaces, there exist unique zero vectors in each, say $\underline{0}_V$ and $\underline{0}_W$, respectively.

Then

$$(\underline{0}_V, \underline{0}_W) + (\underline{A},\underline{B}) = (\underline{0}_V + \underline{A},\ \underline{0}_W + \underline{B}) = (\underline{A},\underline{B})$$

Similarly, $(\underline{A},\underline{B}) + (\underline{0}_V, \underline{0}_W) = (\underline{A},\underline{B})$

So $(\underline{0}_V, \underline{0}_W)$ is the additive identity of $V \oplus W$.

④ Let $(\underline{A}, \underline{B}) \in V \oplus W$. Then $\underline{A} \in V$ and so has an additive inverse $-\underline{A} \in V$. Also, $\underline{B} \in W$ and so has an additive inverse $-\underline{B} \in W$. Note that

$$(\underline{A}, \underline{B}) + (-\underline{A}, -\underline{B}) = (\underline{A} + (-\underline{A}), \underline{B} + (-\underline{B}))$$

$$= (\underline{0}_V, \underline{0}_W)$$

So $(\underline{A}, \underline{B})$ has an additive inverse.

⑤ Multiplicative Identity

$$1(\underline{A}, \underline{B}) = (1\underline{A}, 1\underline{B}) = (\underline{A}, \underline{B})$$

⑥ Distributivity Let $r, s \in \mathbb{F}$. Then

$$(r+s)(\underline{A}, \underline{B}) = ((r+s)\underline{A}, (r+s)\underline{B})$$

$$= (r\underline{A} + s\underline{A}, r\underline{B} + s\underline{B})$$

$$= (r\underline{A}, r\underline{B}) + (s\underline{A}, s\underline{B})$$

$$= r(\underline{A}, \underline{B}) + s(\underline{A}, \underline{B})$$

⑦ Distributivity Let $r \in \mathbb{F}$. Then

$$r((\underline{A}, \underline{B}) + (\underline{C}, \underline{D})) = r(\underline{A} + \underline{C}, \underline{B} + \underline{D})$$

$$= \left(r(\underline{A}+\underline{C}),\, r(\underline{B}+\underline{D})\right)$$

$$= \left(r\underline{A}+r\underline{C},\ r\underline{B}+r\underline{D}\right)$$

$$= (r\underline{A},\, r\underline{B}) + (r\underline{C},\, r\underline{D})$$

$$= r(\underline{A},\underline{B}) + r(\underline{C},\underline{D})$$

8 Let $r, s \in \mathbb{F}$. Then

$$(rs)(\underline{A},\underline{B}) = (rs\underline{A},\, rs\underline{B}) = \left(r(s\underline{A}),\, r(s\underline{B})\right)$$

$$= r\,(s\underline{A},\, s\underline{B})$$

$$= r\left(s(\underline{A},\underline{B})\right)$$

Q4 (a) We first verify that we have closure under addition and scalar multiplication by $\mathbb{R}$.

- Let $x, y \in V$ and $\alpha \in \mathbb{R}$. Then

(i) $x \oplus y = xy \in V$ since $x>0, y>0 \Rightarrow xy>0$
ie V is closed under $\oplus$

(ii) $\alpha \circ x = x^{\alpha} \in V$ since $x^{\alpha} > 0$
ie V is closed under scalar mult. $\circ$

We now verify the axioms of a vector space:

1. <u>Commutativity</u>:
$x \oplus y = xy = yx = y \oplus x$

2. <u>Associativity</u>: Let $z \in V$. Then

$$(x \oplus y) \oplus z = xy \oplus z = xyz = x(yz)$$
$$= x \oplus yz$$
$$= x \oplus (y \oplus z)$$

3. <u>Additive Identity</u>: $1 \in V$ and for any $x \in V$ we have
$1 \oplus x = 1x = x$ and $x \oplus 1 = x1 = x$
$\therefore$ The zero vector of V is 1

4. <u>Additive Inverse</u>: Let $x \in V$. Let $-x = \frac{1}{x} \in V$
Then
$x \oplus (-x) = x(-x) = x\left(\frac{1}{x}\right) = 1$
That is, the additive inverse of x is $\frac{1}{x}$

5. <u>Multiplicative Identity</u>

$1 \circ x = x^1 = x$

6. <u>Distributivity</u> Let $r, s \in \mathbb{R}$. Then

$$(r+s) \circ x = x^{r+s} = x^r x^s = x^r \oplus x^s$$
$$= (r \circ x) \oplus (s \circ x)$$

7 <u>Distributivity</u> Let $r \in \mathbb{R}$. Then

$$r \circ (x \oplus y) = r \circ (xy) = (xy)^r = x^r y^r$$
$$= x^r \oplus y^r$$
$$= (r \circ x) \oplus (r \circ y)$$

8 Let $r, s \in \mathbb{R}$. Then

$$(rs) \circ x = x^{rs} = (x^s)^r = r \circ (x^s)$$
$$= r \circ (s \circ x)$$

b By above, the additive inverse of $\underline{x} = 4$ is $\dfrac{1}{4}$

Q5 Note that V is closed under addition and scalar multiplication:
if $\underline{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $\underline{B} = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \in V$ and $\alpha \in \mathbb{R}$
then

(i) $\underline{A} + \underline{B} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix} \in V$

and

(ii) $\alpha \underline{A} = \begin{bmatrix} \alpha a & 0 \\ 0 & \alpha b \end{bmatrix} \in V$

Checking the 8 axioms of a vector space is similar to that of Q1. We leave the details to the student.

Q6 Let $(a,b), (c,d) \in V$ and $r \in \mathbb{R}$. Then

i $(a,b) \oplus (c,d) = (ad+bc, bd) \in V$ and
ii $r \circ (a,b) = (rab^{r-1}, b^r) \in V$

i.e. V is closed under the addition and scalar multiplication.

Below we partially verify that V is a vector space and leave the remainder of the details to the student.

Additive Identity

Let $\underline{0} = (0,1) \in V$. Then note that if $(a,b) \in V$ we have

$$\underline{0} \oplus (a,b) = (0,1) \oplus (a,b) = (0+a, b) = (a,b)$$

and

$$(a,b) \oplus (0,1) = (a+0, b) = (a,b)$$

So the zero vector is $(0,1)$.

<u>Additive Inverse:</u> Given $(a,b) \in V$ we let

$$-(a,b) = \left(\frac{-a}{b^2}, \frac{1}{b}\right) \in V.$$ Then

$$\begin{aligned}(a,b) \oplus -(a,b) &= (a,b) \oplus \left(\frac{-a}{b^2}, \frac{1}{b}\right)\\ &= \left(a \cdot \frac{1}{b} + b\left(\frac{-a}{b^2}\right), b\left(\frac{1}{b}\right)\right)\\ &= (0,1)\end{aligned}$$

That is, the additive inverse of (a,b) is $\left(\frac{-a}{b^2}, \frac{1}{b}\right)$

<u>Distributivity:</u> Let $r \in \mathbb{R}$. Then

$$\begin{aligned} r \circ ((a,b) \oplus (c,d)) &= r \circ (ad+bc, bd)\\ &= (r(ad+bc)(bd)^{r-1}, (bd)^r)\\ &= (r(ad+bc)b^{r-1}d^{r-1}, b^r d^r)\\ &= (rab^{r-1}d^r + rcd^{r-1}b^r, b^r d^r)\\ &= (rab^{r-1}d^r + b^r rcd^{r-1}, b^r d^r)\\ &= (rab^{r-1}, b^r) \oplus (rcd^{r-1}, d^r)\\ &= [r \circ (a,b)] \oplus [r \circ (c,d)]\end{aligned}$$

Q7 We first verify that V is closed under the given operations.

Let $\underline{x} = \{x_0, x_1, \ldots\}$, $y = \{y_0, y_1, \ldots\} \in V$ and $\alpha \in \mathbb{R}$.

Then

$$\underline{x} + y = \{x_0 + y_0, x_1 + y_1, x_2 + y_2, \ldots\} \quad \text{and}$$

$$x_n + y_n = (x_{n-1} + x_{n-2}) + (y_{n-1} + y_{n-2})$$

$$= (x_{n-1} + y_{n-1}) + (x_{n-2} + y_{n-2})$$

for $n \geq 2$. Thus, $\underline{x} + y \in V$

So V is closed under addition.

Also,

$$\alpha \underline{x} = \{\alpha x_0, \alpha x_1, \alpha x_2, \ldots\} \quad \text{and}$$

$$\alpha x_n = \alpha (x_{n-1} + x_{n-2}) = \alpha x_{n-1} + \alpha x_{n-2}$$

for $n \geq 2$

That is, $\alpha \underline{x} \in V$. and so V is closed under scalar multilplication.

We now partially verify that V is a vector space over $\mathbb{R}$, leaving the remainder of the details to the students.

<u>Commutativity</u>

$$\underline{x} + y = \{x_0 + y_0, x_1 + y_1, \ldots\}$$

$$= \{y_0 + x_0, y_1 + x_1, \ldots\} = y + \underline{x}$$

Additive Identity

$\underline{0} = \{0,0,0,\dots\} \in V$ and

$\underline{0} + \underline{x} = \{0+x_0, 0+x_1, \dots\} = \{x_0, x_1, \dots\} = \underline{x}$

Similarly, $\underline{x} + \underline{0} = \underline{x}$. so $\underline{0}$ is the zero vector in this example.

Additive Inverse

Let $\underline{x} = \{x_0, x_1, x_2, \dots\} \in V$. Let

$$-\underline{x} = \{-x_0, -x_1, -x_2, \dots\}.$$

Observe that for $n \geq 2$, we have

$$-x_n = -(x_{n-1} + x_{n-2}) = -x_{n-1} - x_{n-2}$$

and so $-\underline{x} \in V$. Also,

$$\underline{x} + (-\underline{x}) = \{x_0 - x_0, x_1 - x_1, \dots\} = \{0,0,\dots\}$$
$$= \underline{0}.$$

So $-\underline{x}$ is the additive inverse of $\underline{x}$

Distributivity Let $r, s \in \mathbb{R}$. Then

$$(r+s)\underline{x} = (r+s)\{x_0, x_1, \dots\} = \{(r+s)x_0, (r+s)x_1, \dots\}$$
$$= \{rx_0 + sx_0, rx_1 + sx_1, \dots\}$$

$$= \{rx_0, rx_1, \ldots\} + \{sx_0, sx_1, \ldots\}$$

$$= r\underline{x} + s\underline{x}$$