Tutorial Worksheet #10
Some Sample Solutions

1 (a) We know that if θ is the angle between 1 and x, then

$$\cos(\theta) = \frac{\langle 1, x\rangle}{\|1\|\,\|x\|}$$

Now

$$\langle 1, x\rangle = \int_0^3 x\,dx = \frac{x^2}{2}\Big|_0^3 = \frac{9}{2}$$

$$\|1\| = \sqrt{\int_0^3 1\,dx} = \sqrt{x\Big|_0^3} = \sqrt{3}$$

$$\|x\| = \sqrt{\int_0^3 x^2\,dx} = \sqrt{x^3/3\Big|_0^3} = \sqrt{9} = 3$$

So

$$\cos(\theta) = \frac{\frac{9}{2}}{(\sqrt{3})(3)} = \frac{9}{6\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

So $\theta = \frac{\pi}{6}$ or $\frac{11\pi}{6}$ radians

(30° or 330°)

(b) $$||x^2-1|| = \sqrt{\int_0^3 (x^2-1)^2\,dx}$$

$$= \sqrt{\int_0^3 (x^4-2x^2+1)\,dx}$$

$$= \sqrt{\left(\frac{x^5}{5} - \frac{2}{3}x^3 + x\right)\Big|_0^3}$$

$$= \sqrt{\frac{3^5}{5} - \frac{2}{3}(3)^3 + 3}$$

$$= \sqrt{\frac{168}{5}}$$

(2) We work in $V = \mathbb{R}^n$ with the usual dot product for our inner product.

Let $\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$.

Then $\langle \underline{a}, \underline{b} \rangle = a_1 + \cdots + a_n$

$$||\underline{a}|| = \sqrt{a_1^2 + \cdots + a_n^2}$$

and

$$||\underline{b}|| = \sqrt{\underbrace{1 + \cdots + 1}_{n}} = \sqrt{n}$$

So, by the Cauchy - Schwarz Inequality,

$$|\langle \underline{a}, \underline{b} \rangle| = |a_1 + \cdots + a_n| \leq \|\underline{a}\| \|\underline{b}\|$$
$$= \left(\sqrt{a_1^2 + \cdots + a_n^2}\right)\left(\sqrt{n}\right)$$

Thus,

$$|a_1 + \cdots + a_n|^2 = (a_1 + \cdots + a_n)^2$$
$$\leq n\left(a_1^2 + \cdots + a_n^2\right)$$

3 For sake of notation, let

$\underline{b}_1 = 1$

$\underline{b}_2 = 4x + (-2-i)$

$\underline{b}_3 = 7x^2 + (-6-5i)x + 4i$

The vectors of $\mathcal{B}$ are scalars of $\underline{b}_1, \underline{b}_2, \underline{b}_3$.

Then we have the following data

p	b_1	b_2	b_3
$p(0)$	1	$-2-i$	$4i$
$p(1)$	1	$2-i$	$1-i$
$p(i)$	1	$-2+3i$	$-2-2i$

Thus,

$$\langle \underline{b}_1, \underline{b}_2 \rangle = 1(-2+i) + 2(1)(2+i) + 1(-2-3i) = 0$$

$$\langle \underline{b}_1, \underline{b}_3 \rangle = 1(-4i) + 2(1)(1+i) + (1)(-2+2i) = 0$$

$$\begin{aligned} \langle \underline{b}_2, \underline{b}_3 \rangle &= (-2-i)(-4i) + 2(2-i)(1+i) \\ &\quad + (-2+3i)(-2+2i) \\ &= (8i-4) + (6+2i) + (-2-10i) \\ &= 0 \end{aligned}$$

Hence $\{\underline{b}_1, \underline{b}_2, \underline{b}_3\}$ and hence $\mathcal{B}$ is an orthogonal set.

Further,

$$\|\underline{b}_1\|^2 = \langle \underline{b}_1, \underline{b}_1 \rangle = 1(1) + 2(1)(1) + 1(1) = 4$$

$$\Rightarrow \|\underline{b}_1\| = 2$$

$$\begin{aligned} \|\underline{b}_2\|^2 = \langle \underline{b}_2, \underline{b}_2 \rangle &= (-2-i)(-2+i) + 2(2-i)(2+i) \\ &\quad + (-2+3i)(-2-3i) \\ &= 5 + 2(5) + 13 = 28 \end{aligned}$$

$$\Rightarrow \|\underline{b}_2\| = \sqrt{28}$$

$$\begin{aligned} \|\underline{b}_3\|^2 = \langle \underline{b}_3, \underline{b}_3 \rangle &= 4i(-4i) + (1-i)(1+i)2 \\ &\quad + (-2-2i)(-2+2i) \\ &= 16 + 4 + 8 = 28 \end{aligned}$$

$$\Rightarrow \|\underline{b}_3\| = \sqrt{28}$$

Thus, $\left\{ \frac{1}{2} \underline{b_1}, \frac{1}{\sqrt{28}} \underline{b_2}, \frac{1}{\sqrt{28}} \underline{b_3} \right\} = \mathcal{B}$ is an orthonormal set.

By a theorem from class, since $\mathcal{B}$ consists of non-zero orthogonal vectors, $\mathcal{B}$ is linearly independent.

So, since $\mathcal{B}$ has 3 linearly independent vectors and $\dim(\mathcal{P}_2(\mathbb{C})) = 3$, $\mathcal{B}$ must also be a basis for $\mathcal{P}_2(\mathbb{C})$.

We conclude that $\mathcal{B}$ is an orthonormal basis for $\mathcal{P}_2(\mathbb{C})$.

(4) By definition of an orthonormal basis,

$\underline{w} = t_1 \underline{v_1} + \cdots + t_n \underline{v_n}$ for unique

scalars $t_1, \ldots, t_n$.

Fix j in $\{1, \ldots, n\}$. Then

$$\begin{aligned} \langle \underline{w}, \underline{v_j} \rangle &= \langle t_1 \underline{v_1} + \cdots + t_n \underline{v_n}, \underline{v_j} \rangle \\ &= t_1 \langle \underline{v_1}, \underline{v_j} \rangle + \cdots + t_n \langle \underline{v_n}, \underline{v_j} \rangle \\ &= t_j \langle \underline{v_j}, \underline{v_j} \rangle \quad \left(\text{since } \mathcal{B} \text{ is orthogonal}\right) \end{aligned}$$

$= t_j (1)$ (since each $\underline{v}_j$ is a unit vector)

$= t_j$

Thus, $t_j = \langle \underline{w}, \underline{v}_j \rangle$. Of course, this is true for all j. Hence

$$[\underline{w}]_{\mathcal{B}} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} = \begin{bmatrix} \langle \underline{w}, \underline{v}_1 \rangle \\ \vdots \\ \langle \underline{w}, \underline{v}_n \rangle \end{bmatrix}.$$

5 (a) The claim is trivially true if $\underline{w} = \underline{0}$. So assume $\underline{w} \neq \underline{0}$.

Note first that for each $\underline{w}_j \in \mathcal{B}$ we have

$$\langle \underline{v} - \mathrm{proj}_W(\underline{v}), \underline{w}_j \rangle$$

$$= \langle \underline{v} - \mathrm{proj}_{\underline{w}_1}(\underline{v}) - \cdots - \mathrm{proj}_{\underline{w}_k}(\underline{v}), \underline{w}_j \rangle$$

$$= \langle \underline{v}, \underline{w}_j \rangle - \langle \mathrm{proj}_{\underline{w}_1}(\underline{v}), \underline{w}_j \rangle - \cdots - \langle \mathrm{proj}_{\underline{w}_k}(\underline{v}), \underline{w}_j \rangle$$

Now for $l \neq j$, we have

$$\langle \operatorname{proj}_{\underline{w}_l}(\underline{v}), \underline{w}_j \rangle = \left\langle \frac{\langle \underline{v}, \underline{w}_l \rangle}{\|\underline{w}_l\|^2} \underline{w}_l, \underline{w}_j \right\rangle$$

$$= \frac{\langle \underline{v}, \underline{w}_l \rangle}{\|\underline{w}_l\|^2} \langle \underline{w}_l, \underline{w}_j \rangle$$

$$= 0$$

(since $\underline{w}_l \perp \underline{w}_j$)

and

$$\langle \operatorname{proj}_{\underline{w}_j}(\underline{v}), \underline{w}_j \rangle = \left\langle \frac{\langle \underline{v}, \underline{w}_j \rangle}{\|\underline{w}_j\|^2} \underline{w}_j, \underline{w}_j \right\rangle$$

$$= \frac{\langle \underline{v}, \underline{w}_j \rangle}{\|\underline{w}_j\|^2} \langle \underline{w}_j, \underline{w}_j \rangle$$

$$= \langle \underline{v}, \underline{w}_j \rangle$$

So, $\langle \underline{v} - \operatorname{proj}_W(\underline{v}), \underline{w}_j \rangle = \langle \underline{v}, \underline{w}_j \rangle - \langle \underline{v}, \underline{w}_j \rangle$

$$= 0$$

Now since $\mathcal{B}$ is a basis for W

we have unique scalars $t_1, \ldots, t_k$ such that

$$\underline{w} = t_1 \underline{w}_1 + \cdots + t_k \underline{w}_k$$

Thus,

$$\langle \underline{w}, \underline{v} - \mathrm{proj}_W(\underline{v}) \rangle$$

$$= \langle t_1 \underline{w}_1 + \cdots + t_k \underline{w}_k, \underline{v} - \mathrm{proj}_W(\underline{v}) \rangle$$

$$= t_1 \langle \underline{w}_1, \underline{v} - \mathrm{proj}_W(\underline{v}) \rangle + \cdots + t_k \langle \underline{w}_k, \underline{v} - \mathrm{proj}_W(\underline{v}) \rangle$$

$$= t_1 \overline{\langle \underline{v} - \mathrm{proj}_W(\underline{v}), \underline{w}_1 \rangle} + \cdots + t_k \overline{\langle \underline{v} - \mathrm{proj}_W(\underline{v}), \underline{w}_k \rangle}$$

$$= t_1 (\overline{0}) + \cdots + t_k (\overline{0})$$

$$= 0$$

(b) Let $\underline{w} \in W$.

Note that $\mathrm{proj}_W(\underline{v}) - \underline{w} \in W$

So, by (a) $\underline{v} - \mathrm{proj}_W(\underline{v})$ and $\mathrm{proj}_W(\underline{v}) - \underline{w}$ are orthogonal. So, by The Pythagorean Theorem, we have

$$\|\underline{v}-\underline{w}\|^2 = \|\underline{v} - \text{proj}_W(\underline{v}) + \text{proj}_W(\underline{v}) - \underline{w}\|^2$$

$$= \|\underline{v} - \text{proj}_W(\underline{v})\|^2 + \|\text{proj}_W(\underline{v}) - \underline{w}\|^2$$

$$\geq \|\underline{v} - \text{proj}_W(\underline{v})\|^2$$

and so $\|\underline{v} - \text{proj}_W(\underline{v})\| \leq \|\underline{v} - \underline{w}\|$

(c) By part (b) we see that

$$\|\underline{v} - \underline{w}\| = \|\underline{v} - \text{proj}_W(\underline{v})\|$$

$$\Rightarrow \|\text{proj}_W(\underline{v}) - \underline{w}\| = 0$$

$$\Rightarrow \underline{w} = \text{proj}_W(\underline{v})$$

(b) Let $\mathcal{B} = \{\underline{w}_1, \ldots, \underline{w}_k\}$ be an orthogonal basis for W. Then if $\underline{w} \in W$,

$$\text{proj}_W(\underline{w}) = \text{proj}_{\underline{w}_1}(\underline{w}) + \cdots + \text{proj}_{\underline{w}_k}(\underline{w})$$

$$= \frac{\langle \underline{w}, \underline{w}_1 \rangle}{\|w_1\|^2} \underline{w}_1 + \cdots + \frac{\langle \underline{w}, \underline{w}_k \rangle}{\|\underline{w}_k\|^2} \underline{w}_k$$

Since $\mathcal{B}$ is an orthogonal basis for W we have

$$\underline{w} = t_1 \underline{w}_1 + \cdots + t_k \underline{w}_k$$

for unique scalars $t_1, \ldots, t_k$

Consider

$$\langle \underline{w}, \underline{w}_j \rangle = \langle t_1 \underline{w}_1 + \cdots + t_k \underline{w}_k, \underline{w}_j \rangle$$

$$= t_1 \langle \underline{w}_1, \underline{w}_j \rangle + \cdots + t_k \langle \underline{w}_k, \underline{w}_j \rangle$$

$$= t_j \langle \underline{w}_j, \underline{w}_j \rangle$$

$$\Rightarrow t_j = \frac{\langle \underline{w}, \underline{w}_j \rangle}{\langle \underline{w}_j, \underline{w}_j \rangle} = \frac{\langle \underline{w}, \underline{w}_j \rangle}{\|\underline{w}_j\|^2}$$

So $\underline{w} = t_1 \underline{w}_1 + \cdots + t_k \underline{w}_k$

$$= \frac{\langle \underline{w}, \underline{w}_1 \rangle}{\|\underline{w}_1\|^2} \underline{w}_1 + \cdots + \frac{\langle \underline{w}, \underline{w}_k \rangle}{\|\underline{w}_k\|^2} \underline{w}_k$$

$$= \mathrm{proj}_W(\underline{w})$$

7 We start with the standard basis for $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Let $\underline{w}_1 = \underline{e}_1$

$$\underline{w}_2 = \underline{e}_2 - \frac{\langle \underline{e}_2, \underline{w}_1 \rangle}{\|\underline{w}_1\|^2} \underline{w}_1$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}(-1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{w}_3 = \underline{e}_3 - \frac{\langle \underline{e}_3, \underline{w}_1 \rangle}{\|\underline{w}_1\|^2} \underline{w}_1 - \frac{\langle \underline{e}_3, \underline{w}_2 \rangle}{\|\underline{w}_2\|^2} \underline{w}_2$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{0}{2} \underline{w}_1 - \frac{(-1)}{3/2} \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix}$$

Now $\|\underline{w}_1\| = \sqrt{2}$

$$\|\underline{w}_2\| = \sqrt{\frac{3}{2}}$$

$$\|\underline{w}_3\| = \sqrt{\frac{12}{9}} = \frac{2\sqrt{3}}{3}$$

So we let $\underline{u}_1 = \dfrac{\underline{w}_1}{\|\underline{w}_1\|} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$

$$\underline{u}_2 = \frac{\underline{w}_2}{\|\underline{w}_2\|} = \sqrt{\frac{2}{3}} \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{u}_3 = \frac{\underline{w}_3}{\|\underline{w}_3\|} = \frac{2\sqrt{3}}{3} \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix}$$

Then $\mathcal{D} = \{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ is an orthonormal basis of $\mathbb{R}^3$ with respect to this inner product.

8. (a) Note that if $\underline{w} \in W$, then

$\langle \underline{0}, \underline{w} \rangle = 0$ and so $\underline{0} \in W^{\perp}$

Let $\underline{u}, \underline{v} \in W^{\perp}$ and α be a scalar

Then for all $\underline{w} \in W$ we have

$$\langle \underline{u}+\underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle$$

$$= 0 + 0 = 0$$

$\underline{\text{ie}}$ $\underline{u}+\underline{v} \in W^{\perp}$

and

$$\langle \alpha \underline{u}, \underline{w} \rangle = \alpha \langle \underline{u}, \underline{w} \rangle = \alpha(0) = 0$$

$\underline{\text{ie}}$ $\alpha \underline{u} \in W^{\perp}$

By the Subspace Test, $W^{\perp}$ is a subspace of V.

(b) If $\underline{v} \in W$ <u>and</u> $\underline{v} \in W^{\perp}$

then $\langle \underline{v}, \underline{w} \rangle = 0$ for all $\underline{w} \in W$

implies that in particular

$\langle \underline{v}, \underline{v} \rangle = 0$. Thus, $\underline{v} = 0$.

(c) Suppose

$$s_1 \underline{w}_1 + \cdots + s_k \underline{w}_k + t_1 \underline{u}_1 + \cdots + t_m \underline{u}_m = \underline{0}$$

Then

$$(\star) \quad \underbrace{s_1 \underline{w}_1 + \cdots + s_k \underline{w}_k}_{\in W} = \underbrace{-t_1 \underline{u}_1 - \cdots - t_m \underline{u}_m}_{\in W^\perp}$$

So $(\star)$ is in $W \cap W^\perp$

By (b) we have $s_1 \underline{w}_1 + \cdots + s_k \underline{w}_k = \underline{0}$

and $-t_1 \underline{u}_1 - \cdots - t_m \underline{u}_m = \underline{0}$

Since $\{\underline{w}_1, \ldots, \underline{w}_k\}$ and $\{\underline{u}_1, \ldots, \underline{u}_m\}$ are linearly independent, we have

$$s_1 = \cdots = s_m = t_1 = \cdots = t_m = 0$$

as desired.

(d) By definition of $\text{perp}_W(\underline{v})$ we have

$$\underline{v} = \text{proj}_W(\underline{v}) + \text{perp}_W(\underline{v})$$

Now $\text{proj}_W(\underline{v}) \in W$ (by definition)
and
$\text{perp}_W(\underline{v}) \in W^\perp$ (by #5 (a))

Let $\underline{w}_1 = \text{proj}_W(\underline{v})$

$\underline{w}_2 = \text{perp}_W(\underline{v})$.

(e) Let $\{\underline{w}_1, \dots, \underline{w}_k\}$ be a basis for W and $\{\underline{u}_1, \dots, \underline{u}_m\}$ be a basis for $W^\perp$.

By (c) we know $\mathcal{B} = \{\underline{w}_1, \dots, \underline{w}_k, \underline{u}_1, \dots, \underline{u}_m\}$ is linearly independent in V.

Moreover, by (d), $\text{Span}(\mathcal{B}) = V$

Thus, $\mathcal{B}$ is a basis for V

and so

$$\dim(V) = k + m = \dim(W) + \dim(W^\perp)$$

(9) (a) Let $\underline{v}, \underline{u} \in V$. and α be a scalar

Then

$T(\underline{u}+\underline{v}) = \text{proj}_W(\underline{u}+\underline{v})$

$= \text{proj}_W(\underline{u}) + \text{proj}_W(\underline{v})$

(by definition of proj)

$= T(\underline{u}) + T(\underline{v})$

and

$$T(\alpha \underline{v}) = \text{proj}_W(\alpha \underline{v})$$

$$= \alpha \, \text{proj}_W(\underline{v})$$

(by def$^{\underline{n}}$ of proj and inner products)

$$= \alpha T(\underline{v})$$

(b) $\text{Im}(T) = \{\underline{w} \in V \mid \underline{w} = \text{proj}_W(\underline{v}) \text{ for some } \underline{v} \in V\}$

Now if $\underline{v} \in V$ then

$\text{proj}_W(\underline{v}) \in W$ by definition. ie $\text{Im}(T) \subseteq W$

Conversely, if $\underline{w} \in W$ then $\underline{w} \in \text{Im}(T)$

by #6 since $\underline{w} = \text{proj}_W(\underline{w})$

ie $W \subseteq \text{Im}(T)$

Hence, $\text{Im}(T) = W$

(c) $\text{Ker}(T) = \{\underline{v} \in V \mid \text{proj}_W(\underline{v}) = 0\}$

<u>Claim</u> $\text{Ker}(T) = W^{\perp}$

Let $\underline{x} \in \text{Ker}(T)$. Then $\text{proj}_{W}(\underline{x}) = 0$.

By #5a for all $\underline{w} \in W$ we have

$\underline{w}$ and $\underline{x} - \text{proj}_{W}(x) = \underline{x} - \underline{0} = \underline{x}$

are orthogonal. Thus, $x \in W^{\perp}$
ie $\text{Ker}(T) \subseteq W^{\perp}$

Conversely, let $\underline{v} \in W^{\perp}$. Then

$\langle \underline{v}, \underline{w} \rangle = 0$ for all $\underline{w} \in W$.

Thus, by definition of $\text{proj}_{W}(\underline{v})$, we

have $\text{proj}_{W}(\underline{v}) = 0$ (since all

the scalars in the sum have
numerator equal to 0) and so
$\underline{v} \in \text{Ker}(T)$

ie $W^{\perp} \subseteq \text{Ker}(T)$

$\square$

(d) By the Rank-Nullity Theorem,

$$\dim(V) = \dim(\text{Im}(T)) + \dim(\text{Ker}(T))$$
$$= \dim(W) + \dim(W^{\perp})$$

@

10 Let $\underline{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

If $\underline{w} \in W^{\perp}$ then $\underline{w} \cdot \underline{v}_1 = 0$

$$\underline{w} \cdot \underline{v}_2 = 0$$

$$\underline{w} \cdot \underline{v}_3 = 0$$

Conversely, if $\underline{w} \cdot \underline{v}_1 = \underline{w} \cdot \underline{v}_2 = \underline{w} \cdot \underline{v}_3 = 0$

and $\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \alpha_3 \underline{v}_3 \in W$ then

$$\underline{w} \cdot (\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \alpha_3 \underline{v}_3) = \alpha_1 \underline{w} \cdot \underline{v}_1 + \alpha_2 \underline{w} \cdot \underline{v}_2 + \alpha_3 \underline{w} \cdot \underline{v}_3$$

$$= 0$$

ie $\underline{w} \in W^{\perp}$

That is, $\underline{w} \in W^{\perp}$ if and only if

$$\underline{w} \cdot \underline{v}_1 = \underline{w} \cdot \underline{v}_2 = \underline{w} \cdot \underline{v}_3 = 0$$

Let $\underline{w} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

Then by looking at dot products we have

$w \in W^\perp$ if and only if

$$\begin{aligned} 2a + b - c &= 0 \\ a + 2b + c + d &= 0 \\ -b - c + d &= 0 \end{aligned}$$

Solving this system yields

$$W^\perp = \text{Span}\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}\right)$$

So, $W^\perp$ has basis $\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$

(b) Let $\underline{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\underline{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

Then $\text{proj}_{W^\perp}(\underline{v}) = \text{proj}_{\underline{w}}(\underline{v}) = \dfrac{\underline{v} \cdot \underline{w}}{\|\underline{w}\|^2}\,\underline{w}$

$$= \frac{1}{3}\,\underline{w} = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \\ 0 \end{bmatrix}$$

c) $\text{perp}_{W^\perp}(\underline{v}) = \underline{v} - \text{proj}_{W^\perp}(\underline{v})$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 4/3 \\ 2/3 \\ 1 \end{bmatrix}$$

11) a) Let $\underline{v} \in V$. Then
$\text{perp}_W(\underline{v}) = \underline{v} - \text{proj}_W(\underline{v})$

By 5a $\langle \underline{v} - \text{proj}_W(\underline{v}), \underline{w} \rangle = 0$

for all $\underline{w} \in W$. Thus,

$\underline{v} - \text{proj}_W(\underline{v}) \in W^\perp$ by definition.

($\Rightarrow$)
b) Suppose that $\underline{w} = \text{proj}_W(\underline{v})$. Then

$\underline{v} - \underline{w} = \underline{v} - \text{proj}_W(\underline{v}) = \text{perp}_W(\underline{v}) \in W^\perp$

by part a

($\Leftarrow$) Conversely, suppose that

$\underline{v} - \underline{w} \in W^\perp$

<u>Claim</u> From 8b, $\underline{w}_1$ and $\underline{w}_2$ are unique.

<u>Pf</u> Suppose $\underline{v} = \underline{w}_1 + \underline{w}_2 = \underline{\tilde{w}}_1 + \underline{\tilde{w}}_2$

where $\underline{w}_1, \underline{\tilde{w}}_1 \in W$ and $\underline{w}_2, \underline{\tilde{w}}_2 \in W^\perp$

Then $\underbrace{\underline{w}_1 - \underline{\tilde{w}}_1}_{\in W} = \underbrace{\underline{\tilde{w}}_2 - \underline{w}_2}_{\in W^\perp}$

(since W and $W^\perp$ are subspaces)

Thus, $\underline{w}_1 - \underline{\tilde{w}}_1 \in W \cap W^\perp = \{\underline{0}\}$

<u>ie</u> $\underline{w}_1 - \underline{\tilde{w}}_1 = \underline{0}$

Similarly $\underline{w}_2 = \underline{\tilde{w}}_2$. $\square$

Now we uniquely write

$$\underline{v} = \underline{w} + (\underline{v} - \underline{w})$$

and $\underline{w} = \text{proj}_W(\underline{v})$, $\underline{v} - \underline{w} = \text{perp}_W(\underline{v})$

(see #8d)