MATH 2090, Fall 2018 **Dr. S. Cooper**

Tutorial Worksheet #10

Monday, December 3

Name and Student Number: ______________________________

Work the following exercises. *Explain all of your reasoning.* Remember to use good notation and full sentences. All cited exercises are from the textbook *Linear Algebra* by Larry Smith.

1. Consider $\mathcal{P}_2(\mathbb{R})$ with respect to the inner product
$$\langle p, q\rangle = \int_0^3 p(x)q(x)\,dx.$$
   (a) Compute the angle between 1 and x.
   (b) Compute the norm of $x^2 - 1$.

2. Use the Cauchy-Schwarz Inequality to prove that for all real numbers $a_1, \ldots, a_n$, we have
$$(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2).$$

3. Show that
$$\mathcal{B} = \left\{\frac{1}{2}, \frac{1}{\sqrt{28}}(4x + (-2-i)), \frac{1}{\sqrt{28}}(7x^2 + (-6-5i)x + 4i)\right\}$$
is an orthonormal basis for $\mathcal{P}_2(\mathbb{C})$ with respect to the inner product
$$\langle p, q\rangle = p(0)\overline{q(0)} + 2p(1)\overline{q(1)} + p(i)\overline{q(i)}.$$

4. Let $\mathcal{B} = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ be an orthonormal basis for an inner product space V. Let $\mathbf{w} \in V$. Prove that
$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} \langle \mathbf{w}, \mathbf{v_1}\rangle \\ \vdots \\ \langle \mathbf{w}, \mathbf{v_n}\rangle \end{bmatrix}.$$

5. Let $\mathcal{B} = \{\mathbf{w_1}, \ldots, \mathbf{w_k}\}$ be an orthogonal basis for a subspace W of the inner product space V. Let $\mathbf{v} \in V$.
   (a) Prove that for all $\mathbf{w} \in W$ we have that $\mathbf{w}$ and $\mathbf{v} - \text{proj}_W(\mathbf{v})$ are orthogonal.
   (b) Prove for all $\mathbf{w} \in W$ we have $||\mathbf{v} - \text{proj}_W(\mathbf{v})|| \leq ||\mathbf{v} - \mathbf{w}||$.
   (c) Prove that if $||\mathbf{v} - \text{proj}_W(\mathbf{v})|| = ||\mathbf{v} - \mathbf{w}||$ for any vector $\mathbf{w} \in W$, then $\mathbf{w} = \text{proj}_W(\mathbf{v})$.

6. Let W be subspace of the inner product space V. Prove that if $\mathbf{w} \in W$, then $\text{proj}_W(\mathbf{w}) = \mathbf{w}$.

7. Use the Gram-Schmidt Orthogonalization Procedure to find an orthonormal basis of $\mathbb{R}^3$ with respect to the inner product

$$\left\langle \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right\rangle = 2v_1w_1 + 2v_2w_2 + 2v_3w_3 - v_1w_2 - v_2w_1 - v_2w_3 - v_3w_2.$$

8. Let V be an inner product space and let W be a subspace of V. Recall that the **orthogonal complement** of W is defined to be

$$\{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}.$$

   (a) Prove that $W^\perp$ is a subspace of V.
   (b) Prove that $W \cap W^\perp = \{\mathbf{0}\}$.
   (c) Prove that if $\{\mathbf{w_1}, \ldots, \mathbf{w_k}\} \subseteq W$ and $\{\mathbf{u_1}, \ldots, \mathbf{u_m}\} \subseteq W^\perp$ are linearly independent sets, then $\{\mathbf{w_1}, \ldots, \mathbf{w_k}, \mathbf{u_1}, \ldots, \mathbf{u_m}\}$ is a linearly independent set in V.
   (d) Prove that every vector $\mathbf{v} \in V$ can be written as $\mathbf{v} = \mathbf{w_1} + \mathbf{w_2}$ for some $\mathbf{w_1} \in W$ and $\mathbf{w_2} \in W^\perp$.
   (e) Assume that V is finite-dimensional. Prove that $\dim(W) + \dim(W^\perp) = \dim(V)$.

9. Let W be a subspace of the inner product space V. Define the map $T : V \to V$ by $T(\mathbf{v}) = \text{proj}_W(\mathbf{v})$.

   (a) Prove that T is a linear map.
   (b) What is $\text{Im}(T)$?
   (c) What is $\ker(T)$?
   (d) Assume that V is finite-dimensional. Prove that $\dim(W) + \dim(W^\perp) = \dim V$.

10. Consider $\mathbb{R}^4$ with respect to the usual dot product. Let W be the subspace

$$W = \text{Span}\left(\left\{\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}\right\}\right).$$

   (a) Find a basis for $W^\perp$.
   (b) Compute $\text{proj}_{W^\perp}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$.
   (c) Compute $\text{perp}_{W^\perp}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

11. Let W be subspace of the inner product space V.

   (a) Prove that $\text{perp}_W(\mathbf{v}) \in W^\perp$ for all $\mathbf{v} \in V$.
   (b) Let $\mathbf{w} \in W$ and $\mathbf{v} \in V$. Prove that $\mathbf{w} = \text{proj}_W(\mathbf{v})$ if and only if $\mathbf{v} - \mathbf{w} \in W^\perp$.