Thanksgiving Worksheet
Some Sample Solutions

1) Let $\underline{x}, y \in V$ and $\alpha \in \mathbb{F}$. We have

a) We can uniquely write

$$\underline{x} = a_1 \underline{v_1} + \cdots + a_n \underline{v_n}$$

$$y = b_1 \underline{v_1} + \cdots + b_n \underline{v_n}$$

for scalars $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{F}$.

Thus,

$$\begin{aligned} T(\underline{x}+y) &= T\left((a_1\underline{v_1} + \cdots + a_n \underline{v_n}) + (b_1 \underline{v_1} + \cdots + b_n \underline{v_n})\right) \\ &= T\left((a_1+b_1)\underline{v_1} + \cdots + (a_n+b_n)\underline{v_n}\right) \\ &= (a_1+b_1)\underline{w_1} + \cdots + (a_n+b_n)\underline{w_n} \\ &= (a_1\underline{w_1} + \cdots + a_n\underline{w_n}) + (b_1\underline{w_1} + \cdots + b_n\underline{w_n}) \\ &= T(\underline{x}) + T(y) \end{aligned}$$

and

$$\begin{aligned} T(\alpha\underline{x}) &= T\left(\alpha(a_1\underline{v_1} + \cdots + a_n\underline{v_n})\right) \\ &= T\left(\alpha a_1 \underline{v_1} + \cdots + \alpha a_n \underline{v_n}\right) \\ &= \alpha a_1 \underline{w_1} + \cdots + \alpha a_n \underline{w_n} \\ &= \alpha\left(a_1\underline{w_1} + \cdots + a_n\underline{w_n}\right) \end{aligned}$$

$= \alpha T(\underline{x})$

Thus, by definition, T is a linear transformation.

2. Recall that we define $S+T: V \to W$ by

$$(S+T)(\underline{v}) = S(\underline{v}) + T(\underline{v}) \quad \text{for all } \underline{v} \in V$$

and

$\alpha T : V \to W$ by

$$(\alpha T)(\underline{v}) = \alpha T(\underline{v}) \quad \text{for all } \underline{v} \in V.$$

We first show that $S+T$ is a linear transformation

Let $\underline{x}, y \in V$. Then

$$\begin{aligned}(S+T)(\underline{x}+y) &= S(\underline{x}+y) + T(\underline{x}+y) \\ &= S(\underline{x}) + S(y) + T(\underline{x}) + T(y) \\ &= \left(S(\underline{x}) + T(\underline{x})\right) + \left(S(y) + T(y)\right) \\ &= (S+T)(\underline{x}) + (S+T)(y)\end{aligned}$$

Further, if $\alpha \in \mathbb{F}$, then

$$\begin{aligned}(S+T)(\alpha\underline{x}) &= S(\alpha\underline{x}) + T(\alpha\underline{x}) \\ &= \alpha S(\underline{x}) + \alpha T(\underline{x}) \\ &= \alpha\left[S(\underline{x}) + T(\underline{x})\right]\end{aligned}$$

$$= \alpha\left((S+T)(\underline{x})\right)$$

Hence, $S+T: V \to W$ is a linear transformation.

We now show that αT is a linear transformation. Let $\underline{x}, y \in V$ and $a \in \mathbb{F}$. Then:

$$\begin{aligned}(\alpha T)(\underline{x}+y) &= \alpha\left(T(\underline{x}+y)\right)\\ &= \alpha\left(T(\underline{x}) + T(y)\right)\\ &= \alpha T(\underline{x}) + \alpha T(y)\\ &= (\alpha T)(\underline{x}) + (\alpha T)(y)\end{aligned}$$

and

$$\begin{aligned}(\alpha T)(a\underline{x}) &= \alpha\left(T(a\underline{x})\right)\\ &= \alpha\left(a T(\underline{x})\right)\\ &= \alpha a T(\underline{x})\\ &= a \alpha T(\underline{x})\\ &= a\left(\alpha T(\underline{x})\right)\end{aligned}$$

Thus, $\alpha T: V \to W$ is also a linear transformation.

3 Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker(T)$. Then

$$(a-3b+d)+(-2a+6b+2c+2d)x+(2a-6b+2d)x^2 = 0+0x+0x^2$$

Thus, we have the linear system of equations:

$$a-3b+d=0$$
$$-2a+6b+2c+2d=0$$
$$2a-6b+2d=0$$

We set up the augmented matrix and row-reduce:

$$\left[\begin{array}{cccc|c} 1 & -3 & 0 & 1 & 0 \\ -2 & 6 & 2 & 2 & 0 \\ 2 & -6 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We have 2 "free variables"

Let $b=t$, $d=s$ for some $t,s \in \mathbb{R}$ so that

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3t-s \\ t \\ -2s \\ s \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Thus,

$$\ker(T) = \left\{ \begin{bmatrix} 3t-s & t \\ -2s & s \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$= \left\{ t\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} + s\begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$= \text{Span}\left(\left\{\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}\right\}\right)$$

Now if $x, y \in \mathbb{R}$ such that

$$x\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then $\begin{bmatrix} 3x-y & x \\ -2y & y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow x = y = 0$

and so $\left\{\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}\right\}$ is linearly independent.

Hence, $\left\{\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}\right\}$ is a basis for $\text{Ker}(T)$

By the Rank-Nullity Theorem,

$$\begin{aligned} 4 = \dim(M_{2\times 2}(\mathbb{R})) &= \text{rank}(T) + \text{nullity}(T) \\ &= \dim(\text{Im}(T)) + \dim(\text{Ker}(T)) \\ &= \dim(\text{Im}(T)) + 2 \end{aligned}$$

and so, $\dim(\text{Im}(T)) = 4 - 2 = 2$

4 (a) One polynomial in $\operatorname{Ker}(ev_a)$ is

$$p(x) = x - a$$

Check: $ev_a(x-a) = a - a = 0$

(b) The dimension of $\mathcal{U}_a$ is n. To see this, note that

$$\operatorname{Ker}(ev_a) = \{p(x) \in P_n(\mathbb{C}) : p(a) = 0\} = \mathcal{U}_a$$

We also claim that $\operatorname{Im}(ev_a) = \mathbb{C}$. Indeed, let $t \in \mathbb{C}$. Then take $g(x) \in P_n(\mathbb{C})$ to be the constant polynomial

$g(x) = t$. Then $ev_a(g(x)) = g(a) = t$

ie $t \in \operatorname{Im}(ev_a)$ and so $\operatorname{Im}(ev_a) = \mathbb{C}$

Hence, $\operatorname{rank}(ev_a) = \dim(\operatorname{Im}(ev_a)) = \dim(\mathbb{C}) = 1$.

So, by the Rank-Nullity Theorem,

$$\begin{aligned}
\dim(P_n(\mathbb{C})) &= \operatorname{rank}(ev_a) + \operatorname{nullity}(ev_a) \\
n+1 &= 1 + \operatorname{nullity}(ev_a)
\end{aligned}$$

Thus, $\dim(\mathcal{U}_a) = \dim(\operatorname{Ker}(ev_a))$

$$\begin{aligned}
&= \operatorname{nullity}(ev_a) \\
&= n \quad \square
\end{aligned}$$

5 We first find a basis for $Ker(T)$. We have

$$Ker(T) = \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = x - 3y + 2z = 0\}$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid x = 3y - 2z\}$$
$$= \{(3y-2z, y, z) \mid y, z \in \mathbb{R}\}$$
$$= \{y(3,1,0) + z(-2,0,1) \mid y, z \in \mathbb{R}\}$$
$$= Span(\{(3,1,0), (-2,0,1)\})$$

We now note that if $a, b \in \mathbb{R}$ such that

$$a(3,1,0) + b(-2,0,1) = (0,0,0)$$

then

$$(3a-2b, a, b) = (0,0,0)$$

and so $a = b = 0$

Thus, $\{(3,1,0), (-2,0,1)\}$ is linearly independent
We conclude that

$\{(3,1,0), (-2,0,1)\}$ is a basis for $Ker(T)$ and so $\dim(Ker(T)) = nullity(T) = 2$

Hence, by the Rank-Nullity Theorem,

$$3 = \dim(\mathbb{R}^3) = \dim(Ker(T)) + \dim(Im(T))$$
$$= 2 + \dim(Im(T))$$

and so we conclude that

$$\dim(Im(T)) = 3 - 2 = 1.$$

(6) We have

$$\mathrm{Ker}(T) = \{p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \mid T(p) = a_1 + a_2x + a_3x^2 = 0\}$$

$$= \{p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in P_3(\mathbb{R}) \mid a_1 = a_2 = a_3 = 0\}$$

$$= \{p(x) = a_0 \mid a_0 \in \mathbb{R}\}$$

So, $\mathrm{Ker}(T)$ is the set of all constant polynomials.

Now $\mathrm{Ker}(T) = \mathrm{Span}(\{1\})$

and $\{1\}$ is clearly linearly independent

(since $a(1) = 0 \Rightarrow a = 0$). Thus $\{1\}$ is a basis for $\mathrm{Ker}(T)$ and so

$$\dim(\mathrm{Ker}(T)) = \mathrm{nullity}(T) = 1$$

Let $q(x) = a_1 + a_2x + a_3x^2 \in P_2(\mathbb{R})$ be any element in $P_2(\mathbb{R})$

Take $p(x) = s + a_1x + a_2x^2 + a_3x^3$ for any $s \in \mathbb{R}$

Then $T(p(x)) = q(x)$ so that $\mathrm{Im}(T) = P_2(\mathbb{R})$

The standard basis $\{1, x, x^2\}$ for $P_2(\mathbb{R})$ is then also a basis for $\mathrm{Im}(T)$ and so $\dim(\mathrm{Im}(T)) = \mathrm{rank}(T) = 3$.

7 Let V, W be vector spaces with
$\dim(V) = \dim(W)$

($\Rightarrow$) Suppose that $T: V \to W$ is an injective linear transformation. Then, by a theorem from class, $\mathrm{Ker}(T) = \{\underline{0}_V\}$. and so

$$\dim(\mathrm{Ker}(T)) = \mathrm{nullity}(T) = 0.$$

So, by the Rank-Nullity Theorem,

$$\begin{aligned}\dim(V) = n &= \mathrm{Rank}(T) + \mathrm{Nullity}(T) \\ &= \mathrm{Rank}(T) + 0 = \mathrm{Rank}(T)\end{aligned}$$

Thus, $\mathrm{Rank}(T) = \dim(\mathrm{Im}(T)) = n = \dim(W)$

Since $\mathrm{Im}(T)$ is a subspace of W having the same dimension of W, it must be the case that
$\mathrm{Im}(T) = W$ (by a theorem from class).
Thus, T is surjective.

($\Leftarrow$) Conversely, assume that $T: V \to W$ is a surjective linear transformation.
Then $\mathrm{Im}(T) = W$ so that
$\dim(\mathrm{Im}(T)) = \mathrm{Rank}(T) = \dim(W) = n$.
By the Rank-Nullity Theorem, we have

$$\begin{aligned}n = \dim(V) &= \mathrm{Rank}(T) + \mathrm{Nullity}(T) \\ &= n + \mathrm{Nullity}(T)\end{aligned}$$

and so

$$\text{Nullity}(T) = \dim\left(\text{Ker}(T)\right) = 0.$$

Hence, $\text{Ker}(T) = \{\underline{0}_V\}$. So, by a theorem from class, T must be injective as desired $\boxtimes$

8 ($\Rightarrow$) Suppose first that $T: V \to W$ is an injective linear transformation. Then, by a theorem from class, $\ker(T) = \{\underline{0}_V\}$.

Let $a_1, \ldots, a_n \in \mathbb{F}$ be scalars and assume that $a_1 T(\underline{v}_1) + \cdots + a_n T(\underline{v}_n) = \underline{0}_W$.

Then, since T is a linear transformation,

$$T(a_1\underline{v}_1 + \cdots + a_n\underline{v}_n) = \underline{0}_W, \text{ and so}$$

$$a_1\underline{v}_1 + \cdots + a_n\underline{v}_n \in \ker(T) = \{\underline{0}_V\}.$$

That is, $a_1\underline{v}_1 + \cdots + a_n\underline{v}_n = \underline{0}_V$.

But, since $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is a basis for V, $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is linearly independent. Hence, we must have $a_1 = \cdots = a_n = 0$ and thus $\{T(\underline{v}_1), \ldots, T(\underline{v}_n)\}$ is linearly independent.

($\Leftarrow$) Conversely, suppose that $\{T(\underline{v}_1), \ldots, T(\underline{v}_n)\}$ is linearly independent.

Let $\underline{x} \in \ker(T)$. Then $T(\underline{x}) = \underline{0}_W$.

Now, since $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is a basis for V, there are unique scalars $b_1, \ldots, b_n$ such that

$$\underline{x} = b_1\underline{v}_1 + \cdots + b_n\underline{v}_n$$

Thus,

$$\underline{0}_W = T(\underline{x}) = T(b_1\underline{v}_1 + \cdots + b_n\underline{v}_n)$$

$$= b_1 T(\underline{v}_1) + \cdots + b_n T(\underline{v}_n)$$

Since $\{T(\underline{v}_1), \ldots, T(\underline{v}_n)\}$ is linearly independent we must have $b_1 = \cdots = b_n = 0$ and so $\underline{x} = \underline{0}_V$. We conclude that $\ker(T) = \{\underline{0}_V\}$ and hence T must be injective. $\boxtimes$

9. $T(\underline{e}_j) = \underline{a}_j$ for $j = 1, 2, 3$ where

$\underline{e}_1 = (1,0,0)$, $\underline{e}_2 = (0,1,0)$, $\underline{e}_3 = (0,0,1)$
$\underline{a}_1 = (0,1,1)$, $\underline{a}_2 = (-1,0,1)$, $\underline{a}_3 = (0,1,2)$

By definition of a linear extension we have for any $\underline{x} = (x_1, x_2, x_3)$
$= x_1\underline{e}_1 + x_2\underline{e}_2 + x_3\underline{e}_3 \in \mathbb{R}$
that

$$T(\underline{x}) = x_1 T(\underline{e}_1) + x_2 T(\underline{e}_2) + x_3 T(\underline{e}_3)$$

$$= x_1\underline{a}_1 + x_2\underline{a}_2 + x_3\underline{a}_3$$

$$= x_1(0,1,1) + x_2(-1,0,1) + x_3(0,1,2)$$

$$= (-x_2, x_1 + x_3, x_1 + x_2 + 2x_3)$$

So,

$$T(1,2,3) = (-2, 4, 9)$$

Also,

$$\begin{aligned}
\mathrm{Ker}(T) &= \{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid T(x_1,x_2,x_3)=(0,0,0)\} \\
&= \{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid -x_2 = x_1+x_3 = x_1+x_2+2x_3 = 0\} \\
&= \{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_2=0,\ x_1=-x_3=-2x_3\} \\
&= \{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_2=0,\ x_1=-x_3,\ x_3=2x_3\} \\
&= \{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_2=0,\ x_1=-x_3,\ x_3=0\} \\
&= \{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1=x_2=x_3=0\} \\
&= \{(0,0,0)\} = \{\underline{0}_{\mathbb{R}^3}\}
\end{aligned}$$

Thus, T is injective. Hence, by Exercise #7 on this worksheet, T is also surjective. This means that

$$\mathrm{Im}(T) = \mathbb{R}^3$$

(10) We show that T is injective and surjective, and hence T is an isomorphism.

<u>Injective</u>

Let $p(x) \in \text{Ker}(T)$. Then $p(x) = \alpha + \beta x + \gamma x^2$ for some $\alpha, \beta, \gamma \in \mathbb{R}$ and

$$T(p(x)) = \begin{bmatrix} p(a) \\ p(b) \\ p(c) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This means that $p(x)$ has 3 distinct roots in $\mathbb{R}$. But a degree ≤ 2 polynomial can have at most 2 distinct real roots, unless it is the zero polynomial. Hence, $p(x) = \underline{0} = 0 + 0x + 0x^2$

We conclude that

$$\text{Ker}(T) = \{\underline{0}_{P_2(\mathbb{R})}\}$$

and hence T is injective by a theorem from class.

<u>Surjective</u> We show 2 ways to conclude that T is surjective (students need only show 1 way - the extra path is intended for more examples).

<u>Method 1</u>: By Exercise 7 of this worksheet, T is surjective since T is injective and

$$\dim(P_2(\mathbb{R})) = \dim(\mathbb{R}^3) = 3.$$

Method 2: Here we use the definition of surjective and show that $Im(T) = \mathbb{R}^3$.

Let $\begin{bmatrix} s \\ t \\ u \end{bmatrix}$ be any vector in $\mathbb{R}^3$. We want to find $p(x) = \alpha + \beta x + \gamma x^2 \in P_2(\mathbb{R})$ such that

$$T(p(x)) = \begin{bmatrix} p(a) \\ p(b) \\ p(c) \end{bmatrix} = \begin{bmatrix} s \\ t \\ u \end{bmatrix}$$

This gives a linear system of equations in the unknowns α, β, γ:

$$p(a) = \alpha + \beta a + \gamma a^2 = s$$
$$p(b) = \alpha + \beta b + \gamma b^2 = t$$
$$p(c) = \alpha + \beta c + \gamma c^2 = u$$

We set up our augmented matrix and solve, keeping in mind that a, b, c are distinct:

$$\begin{array}{ccc} \alpha & \beta & \gamma \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & a & a^2 & s \\ 1 & b & b^2 & t \\ 1 & c & c^2 & u \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & a & a^2 & s \\ 0 & b-a & b^2-a^2 & t-s \\ 0 & c-a & c^2-a^2 & u-s \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & a & a^2 & s \\ 0 & 1 & b+a & (t-s)/(b-a) \\ 0 & 1 & c+a & (u-s)/(c-a) \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & a & a^2 & s \\ 0 & 1 & b+a & (t-s)/(b-a) \\ 0 & 0 & c-b & (u-s)/(c-a) - (t-s)/(b-a) \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & a & a^2 & s \\ 0 & 1 & b+a & (t-s)/(b-a) \\ 0 & 0 & 1 & 1/(c-b)\left[(u-s)/(c-a) - (t-s)/(b-a)\right] \end{array}\right]$$

We see that the system has 1 unique solution:

$$\gamma \cancel{\beta} = \left[\frac{(u-s)}{(c-a)} - \frac{(t-s)}{(b-a)}\right]\frac{1}{(c-b)}$$

$$= \frac{(u-s)(b-a) - (t-s)(c-a)}{(c-a)(b-a)(c-b)}$$

$$\beta = -(b+a)\gamma + \frac{(t-s)}{(b-a)}$$

$$= \frac{-\gamma(b^2-a^2) + (t-s)}{(b-a)}$$

$$\alpha = -a\beta - a^2\gamma + s$$

So $p(x) = \alpha + \beta x + \gamma x^2$ with α, β, γ as above yields
$T(p(x)) = \begin{bmatrix} s \\ t \\ u \end{bmatrix}$. We conclude that $\mathrm{Im}(T) = \mathbb{R}^3$ so that T is surjective.

(11) If there does not exist a vector $\underline{x}$ not in $\mathrm{Im}(T)$, then we would have $\mathrm{Im}(T) = M_{2\times 2}(\mathbb{C})$. This implies that

$$\dim(\mathrm{Im}(T)) = \mathrm{rank}(T) = \dim(M_{2\times 2}(\mathbb{C}))$$
$$= 4.$$

But, by the Rank-Nullity Theorem, we have

$$\dim(\mathbb{C}^3) = 3 = \mathrm{Rank}(T) + \mathrm{Nullity}(T)$$

Since, $\mathrm{Rank}(T) \geq 0$ and $\mathrm{Nullity}(T) \geq 0$

this says that $\mathrm{Rank}(T) \leq 3$ which is a contradiction to $\mathrm{Rank}(T) = 4$.

Thus, we must have a vector $\underline{x} \in M_{2\times 2}(\mathbb{C})$ such that $\underline{x}$ is not in the image of T.

12 Recall that the direct sum of vector spaces V and W, denoted $V \oplus W$, is the set of all ordered pairs $(\underline{v}, \underline{w})$ where $\underline{v} \in V$ and $\underline{w} \in W$ (see Tutorial Worksheet #1, Exercise 3).

Let $\mathcal{B} = \{\underline{v_1}, \ldots, \underline{v_n}\}$ be a basis for V and
$\mathcal{C} = \{\underline{w_1}, \ldots, \underline{w_m}\}$ be a basis for W.

<u>Claim</u> $\mathcal{D} = \{(\underline{v_1}, \underline{0}_W), \ldots, (\underline{v_n}, \underline{0}_W), (\underline{0}_V, \underline{w_1}), \ldots, (\underline{0}_V, \underline{w_m})\}$ is a basis for $V \oplus W$

<u>Pf</u> Let $(\underline{x}, \underline{y}) \in V \oplus W$. Then we have unique scalars $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ such that

$$\underline{x} = a_1 \underline{v_1} + \cdots + a_n \underline{v_n} \in V \quad \text{and}$$

$$\underline{y} = b_1 \underline{w_1} + \cdots + b_m \underline{w_m} \in W. \quad \text{Thus,}$$

$$(\underline{x}, \underline{y}) = (a_1 \underline{v_1} + \cdots + a_n \underline{v_n}, b_1 \underline{w_1} + \cdots + b_m \underline{w_m})$$

$$= a_1 (\underline{v_1}, \underline{0}_W) + \cdots + a_n (\underline{v_n}, \underline{0}_W) + b_1 (\underline{0}_V, \underline{w_1}) + \cdots + b_m (\underline{0}_V, \underline{w_m})$$

$$\in \operatorname{Span}(\mathcal{D})$$

Thus, $\operatorname{Span}(\mathcal{D}) = V \oplus W$.

In addition, if $\alpha_1, \ldots, \alpha_{n+m}$ are scalars such that

$$\alpha_1(\underline{v}_1, \underline{0}_W) + \cdots + \alpha_n(\underline{v}_n, \underline{0}_W) + \alpha_{n+1}(\underline{0}_V, \underline{w}_1) + \cdots + \alpha_{n+m}(\underline{0}_V, \underline{w}_m)$$

$$= (\underline{0}_V, \underline{0}_W) = \text{zero vector of } V \oplus W$$

Then equating 1^{st} and 2^{nd} coordinates gives

$$\alpha_1 \underline{v}_1 + \cdots + \alpha_n \underline{v}_n = \underline{0}_V \quad \text{and} \quad \alpha_{n+1} \underline{w}_1 + \cdots + \alpha_{n+m} \underline{w}_m = \underline{0}_W$$

Since $\mathcal{B}$ and $\mathcal{C}$ are linearly independent in V and W, respectively, we must have $\alpha_1 = \cdots = \alpha_n = 0$ and $\alpha_{n+1} = \cdots = \alpha_{n+m} = 0$

Thus, $\mathcal{D}$ is linearly independent.

Therefore, $\mathcal{D}$ is a basis for V $\square$

Conclusion: If V and W are finite-dimensional vector spaces over the field $\mathbb{F}$ with $\dim(V) = n$, $\dim(W) = m$, then

$$\dim(V \oplus W) = n + m$$

Therefore, if we let $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$

then $\dim(\mathbb{C}^n \oplus \mathbb{C}^m) = n + m$.

We also know that $\dim(\mathbb{C}^{n+m}) = n + m$

Since, $\dim(\mathbb{C}^n \oplus \mathbb{C}^m) = \dim(\mathbb{C}^{n+m})$ we conclude that $\mathbb{C}^{n+m}$ is isomorphic to $\mathbb{C}^n \oplus \mathbb{C}^m$ (by a theorem from class)