MATH 2090, Fall 2018 **Dr. S. Cooper**

Thanksgiving Worksheet

Name and Student Number: ______________________________

Work the following exercises. *Explain all of your reasoning.* Remember to use good notation and full sentences. All cited exercises are from the textbook *Linear Algebra* by Larry Smith.

1. Let V and W be vector spaces over the field $\mathbb{F}$. Let $\mathcal{B} = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ be an ordered basis for V and $\{\mathbf{w_1}, \ldots, \mathbf{w_n}\}$ be an ordered set of vectors in W. Prove that the linear extension of $T(\mathbf{v_j}) = \mathbf{w_j}$ for $j = 1, \ldots, n$ is a linear transformation $T : V \to W$.

2. Let $S, T : V \to W$ be linear transformations between vector spaces V and W over the field $\mathbb{F}$. Prove that $S + T$ and αT (for any scalar α) are linear transformations.

3. Define the linear transformation $T : M_{2\times 2}(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a - 3b + d) + (-2a + 6b + 2c + 2d)x + (2a - 6b + 2d)x^2.$$

   Find a basis for $\ker(T)$. Without finding $\mathrm{Im}(T)$, determine $\dim(\mathrm{Im}(T))$.

4. For $a \in \mathbb{C}$, define the linear transformation $\mathrm{ev_a} : \mathcal{P}_n(\mathbb{C}) \to \mathbb{C}$ by

$$\mathrm{ev}_a(p(x)) = p(a).$$

   (a) Find one polynomial in the kernel of ev_a.

   (b) For any a, the set of polynomials $\mathcal{U}_a = \{p(x) \in \mathcal{P}_n(\mathbb{C}) : p(a) = 0\}$ is a subspace of $\mathcal{P}_n(\mathbb{C})$. What is $\dim(\mathcal{U}_a)$?

5. (Section 8.7 #3) Let $T : \mathbb{R}^3 \to \mathbb{R}$ be defined by $T(x, y, z) = x - 3y + 2z$. Show that T is linear. Find a basis for the kernel of T. What are $\dim(\ker(T))$ and $\dim(\mathrm{Im}(T))$?

6. (Section 8.7 #5) Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be the linear transformation defined by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + a_2x + a_3x^2.$$

   Find a basis for $\ker(T)$ and $\mathrm{Im}(T)$? What are their dimensions?

7. (Section 8.7 #18) Let V and W be finite-dimensional vector spaces with the same dimension and $T : V \to W$ a linear transformation. Show that T is injective if and only if it is surjective.

8. (Section 8.7 #19) Let $T : V \to W$ be a linear transformation and $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ a basis for V. Show that T is injective if and only if $\{T(\mathbf{v_1}), \ldots, T(\mathbf{v_n})\}$ are linearly independent.

9. (Section 8.7 #22) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear extension of $T(\mathbf{e_j}) = \mathbf{a_j}$, $j = 1, 2, 3$ where $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is the standard basis for $\mathbb{R}^3$ and $\mathbf{a_1} = (0, 1, 1), \mathbf{a_2} = (-1, 0, 1), \mathbf{a_3} = (0, 1, 2)$. Find $\mathrm{Im}(T)$ and $\ker(T)$. Also find $T(1, 2, 3)$.

10. Let a, b and c be distinct real numbers. Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation defined by

$$T(p(x)) = \begin{bmatrix} p(a) \\ p(b) \\ p(c) \end{bmatrix}.$$

Prove that T is an isomorphism.

11. Let $T : \mathbb{C}^3 \to M_{2\times 2}(\mathbb{C})$ be a linear transformation. Prove that there exists a vector $\mathbf{x} \in M_{2\times 2}(\mathbb{C})$ such that $\mathbf{x}$ is not in the image of T.

12. Prove that for positive integers n and m, $\mathbb{C}^{n+m}$ is isomorphic to the direct sum of $\mathbb{C}^n$ and $\mathbb{C}^m$.