MATH 2090: Linear Algebra 2 Dr. S. Cooper, Fall 2018

Mastery Quiz 5 (B02 & B03)

Name and Student Number: Solutions

In the space provided, please write your solutions to the following exercises. *Fully explain your work.* Remember to use good notation and full sentences. No resources (such as notes, texts, cell phones, calculators, translators, etc.) are permitted.

Good Luck!

1. Are the following statements true (T) or false (F)? Circle your answers. No justification is required and no partial credit will be given for each statement. [5 pts]

   (a) The vector $\begin{bmatrix} 1+i \\ 0 \end{bmatrix}$ in $\mathbb{C}^2$ is a unit vector with respect to the standard Hermitian inner product. T F

   (b) In an inner product space, $||-2\mathbf{v}|| = 2||\mathbf{v}||$. T F

   (c) In an inner product space, if $|\langle \mathbf{v} + \mathbf{w} \rangle| = 6$, then $||\mathbf{v}||\ ||\mathbf{w}|| \leq 6$. T F

   (d) Every inner product space has an orthonormal basis. T F

   (e) $\{0, x^2\}$ is an orthogonal set in $\mathcal{P}_3(\mathbb{C})$ with respect to every inner product. T F

2. Let V be an inner product space.

(a) If $\langle \mathbf{v}, \mathbf{w} \rangle = 10 - i$, then what is $\langle \mathbf{w}, \mathbf{v} \rangle$? [1 pt]

$\langle \underline{w}, \underline{v} \rangle = \overline{\langle \underline{v}, \underline{w} \rangle} = 10 + i$

(b) Let $V = M_{2\times 2}(\mathbb{R})$ with respect to the inner product

$$\langle A, B \rangle = \mathrm{tr}(A^T B).$$

Find the norm of $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. Show all of your work. [2 pts]

Let $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. Then

$$\|A\| = \sqrt{\langle A, A \rangle} = \sqrt{\mathrm{tr}(A^T A)} = \sqrt{\mathrm{tr}\left(\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}\right)}$$
$$= \sqrt{8+8}$$
$$= 4$$

(c) Let $V = \mathcal{P}_2(\mathbb{R})$. Suppose $\langle 1, p \rangle = 3$, $\langle x, p \rangle = 0$ and $\langle x^2, p \rangle = -1$ for some $p \in \mathcal{P}_2(\mathbb{R})$. What is $\langle 2 - 4x + 2x^2, p \rangle$? Show all of your work. [3 pts]

$$\langle 2 - 4x + 2x^2, p \rangle = 2\langle 1, p \rangle - 4\langle x, p \rangle + 2\langle x^2, p \rangle$$
$$= 2(3) - 4(0) + 2(-1)$$
$$= 6 - 0 - 2$$
$$= 4$$

3. Show that

$$\left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle = ab - cd$$

is not an inner product on $\mathbb{R}^2$. [2 pts]

$$\left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle = (1)(1) - (1)(1) = 0$$

However, $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So this is not an inner product.

4. Let $\mathbf{v} = \begin{bmatrix} 7 \\ 2-i \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ be in $\mathbb{C}^3$ with resepct to the standard Hermitian inner product.

(a) Find $\text{proj}_{\mathbf{w}}(\mathbf{v})$. [2 pts]

$$\text{proj}_{\underline{w}}(\underline{v}) = \frac{\langle \underline{v}, \underline{w} \rangle}{\|\underline{w}\|^2} \underline{w} = \frac{8}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$$

(b) Find $\text{perp}_{\mathbf{w}}(\mathbf{v})$. [2 pts]

$$\text{perp}_{\underline{w}}(\underline{v}) = \underline{v} - \text{proj}_{\underline{w}}(\underline{v})$$

$$= \begin{bmatrix} 7 \\ 2-i \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2-i \\ -3 \end{bmatrix}$$

5. Let $\mathcal{B} = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ be an orthonormal basis for an inner product space V. Let $\mathbf{w} \in V$. Prove that

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} \langle \mathbf{w}, \mathbf{v_1} \rangle \\ \vdots \\ \langle \mathbf{w}, \mathbf{v_n} \rangle \end{bmatrix}.$$

[3 pts]

Proof Since $\mathcal{B}$ is a basis for V we can find unique scalars $t_1, \ldots, t_n$ such that

$$\underline{w} = t_1 \underline{v_1} + \cdots + t_n \underline{v_n}$$

Now for any $\underline{v_j}$ we have

$$\langle \underline{w}, \underline{v_j} \rangle = \langle t_1 \underline{v_1} + \cdots + t_n \underline{v_n}, \underline{v_j} \rangle$$

$$= t_1 \langle \underline{v_1}, \underline{v_j} \rangle + \cdots + t_n \langle \underline{v_n}, \underline{v_j} \rangle$$

$$= t_j \langle \underline{v_j}, \underline{v_j} \rangle \quad \text{(since } \mathcal{B} \text{ is an orthogonal set)}$$

$$= t_j \quad \text{(since } \mathcal{B} \text{ is an orthonormal set)}$$

Thus, by definition,

$$[\underline{w}]_{\mathcal{B}} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} = \begin{bmatrix} \langle \underline{w}, \underline{v_1} \rangle \\ \vdots \\ \langle \underline{w}, \underline{v_n} \rangle \end{bmatrix}$$