MATH 2090: Linear Algebra 2 Dr. S. Cooper, Fall 2018

Mastery Quiz 4 (B02 & B03)

Name and Student Number: Solutions

In the space provided, please write your solutions to the following exercises. *Fully explain your work.* Remember to use good notation and full sentences. No resources (such as notes, texts, cell phones, calculators, translators, etc.) are permitted.

Good Luck!

1. Are the following statements true (T) or false (F)? Circle your answers. No justification is required and no partial credit will be given for each statement. [5 pts]

   (a) A 2×2 matrix with characteristic polynomial $\lambda^2 + \lambda$ is diagonalizable. T F

   (b) If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of a matrix A corresponding to distinct eigenvalues, then $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. T F

   (c) There exist similar matrices A and B such that A is invertible while B is not invertible. T F

   (d) If a matrix is similar to $D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, then it is diagonalizable. T F

   (e) If 0 is not an eigenvalue of a square matrix A, then A is invertible. T F

2. Two eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 2 \\ -1 & 5 & 6 \end{bmatrix}$ are $\lambda_1 = 1$ and $\lambda_2 = 2$. The eigenspace E_{λ_1} has basis $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ and the eigenspace E_{λ_2} has basis $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\}$.

(a) Find all the eigenvalues of A. [3 pts]

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -1 & 3-\lambda & 2 \\ -1 & 5 & 6-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 5 & 6-\lambda \end{vmatrix}$$

$$= (2-\lambda)[18-9\lambda+\lambda^2-10]$$

$$= (2-\lambda)(\lambda-8)(\lambda-1)$$

So the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 8$

(b) For each eigenvalue of A not equal to 1 or 2, find a basis for the corresponding eigenspace. [4 pts]

$E_{\lambda_3} = \text{Null}(A-8I)$

$$[A-8I \mid \underline{0}] = \left[\begin{array}{ccc|c} -6 & 0 & 0 & 0 \\ -1 & -5 & 2 & 0 \\ -1 & 5 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & -5 & 2 & 0 \\ -1 & 5 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 5 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = t \Rightarrow x_2 = 2/5\,t$ and $x_1 = 0$

So E_{λ_3} has basis $\left\{ \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \right\}$

(c) Find an invertible matrix P and a diagonal matrix D where $P^{-1}AP = D$. [2 pts]

$$P = \begin{bmatrix} 0 & 3 & 0 \\ -1 & -1 & 2 \\ 1 & 2 & 5 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

3. Let A and B be $n \times n$ matrices such that A is similar to B.

(a) Complete the following definition: [2 pts]

A is *similar* to B if $P^{-1}AP = B$ for some invertible $n \times n$ matrix P.

(b) Prove that if B is similar to an $n \times n$ matrix C, then A is similar to C. [2 pts]

By definition, there are $n \times n$ invertible matrices P and Q such that $P^{-1}AP = B$ and $Q^{-1}BQ = C$.

Thus,

$$C = Q^{-1}BQ = Q^{-1}(P^{-1}AP)Q$$
$$= (Q^{-1}P^{-1})A(PQ) = (PQ)^{-1}A(PQ)$$

Since PQ is an invertible matrix, A is similar to C by definition.

(c) Prove that A^3 is similar to B^3. [2 pts]

We have $P^{-1}AP = B$ for some invertible matrix P. Thus,

$$B^3 = (P^{-1}AP)(P^{-1}AP)(P^{-1}AP)$$
$$= P^{-1}A(PP^{-1})A(PP^{-1})AP$$
$$= P^{-1}AAAP$$
$$= P^{-1}A^3P.$$

By definition, A^3 is similar to B^3.