MATH 2090: Linear Algebra 2 Dr. S. Cooper, Fall 2018

Mastery Quiz 3 (B01)

Name and Student Number: Solutions

In the space provided, please write your solutions to the following exercises. *Fully explain your work.* Remember to use good notation and full sentences. No resources (such as notes, texts, cell phones, calculators, translators, etc.) are permitted.

Good Luck!

1. Are the following statements true (T) or false (F)? Circle your answers. No justification is required and no partial credit will be given for each statement. [5 pts]

   (a) Let $\mathcal{B} = \{\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}, \mathbf{M_4}\}$ be the standard basis for $M_{2\times 2}(\mathbb{R})$ and $\mathcal{C} = \{1, x, x^2\}$ be the standard basis for $\mathcal{P}_2(\mathbb{R})$. If a linear transformation $T : M_{2\times 2}(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ has the matrix representation

   $$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 1 & -3 & 0 & 1 \\ -2 & 6 & 2 & 2 \\ 2 & -6 & 0 & 2 \end{bmatrix},$$

   then $T(\mathbf{M_1}) = -3 + 6x - 6x^2$. T F

   (b) If $S : \mathbb{R}^2 \to \mathbb{R}^3$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ are linear transformations, then any matrix representation of $S + T$ has size 2×3. T F

   (c) Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of an n-dimensional vector space. Then the inverse of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is $P_{\mathcal{B}\leftarrow\mathcal{C}}$. T F

   (d) Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of an n-dimensional vector space. If $\mathbf{x} \in V$, then $[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$. T F

   (e) Let V be an n-dimensional vector space and $T : V \to V$ be an isomorphism. Then for any bases $\mathcal{B}$ and $\mathcal{C}$ of V, we have that $[T]_{\mathcal{B}}^{\mathcal{C}}$ is invertible. T F

2. Let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1, x, x^2, x^3\}$ be the standard bases of $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{P}_3(\mathbb{R})$, respectively. Find the matrix of the linear transformation $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ defined by

$$T(p(x)) = xp(x)$$

relative to $\mathcal{B}$ and $\mathcal{C}$. [5 pts]

$$T(1) = x \Rightarrow [T(1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x) = x^2 \Rightarrow [T(x)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(x^2) = x^3 \Rightarrow [T(x^2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus,

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^7$ be the linear transformation whose matrix relative to the standard bases is

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 4 \\ 2 & 0 & 2 \\ 0 & -1 & 3 \\ 3 & 2 & -5 \\ 0 & 0 & 7 \\ 4 & 0 & 1 \end{bmatrix}$$

and let $S : \mathbb{R}^7 \to \mathbb{R}^2$ be the linear transformation whose matrix relative to the standard bases is

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the matrix of $S \circ T : \mathbb{R}^3 \to \mathbb{R}^2$ relative to the standard bases. [3 pts]

From a theorem in class, we have

$$[S \circ T]_{\mathcal{B}}^{\mathcal{C}} = BA$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 4 \\ 2 & 0 & 2 \\ 0 & -1 & 3 \\ 3 & 2 & -5 \\ 0 & 0 & 7 \\ 4 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

where $\mathcal{B}$ = standard basis of $\mathbb{R}^3$
$\mathcal{C}$ = standard basis of $\mathbb{R}^2$

4. Consider the bases $\mathcal{B} = \{1, x, \frac{3}{2}x^2 - \frac{1}{2}, \frac{5}{2}x^3 - \frac{3}{2}x\}$ and let $\mathcal{C} = \{1, x, x^2, x^3\}$ for $\mathcal{P}_3(\mathbb{C})$.

(a) Note that

$$x^2 = \frac{1}{3}(1) + 0x + \frac{2}{3}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + 0\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)$$
$$x^3 = (0)(1) + \frac{3}{5}x + 0\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + \frac{2}{5}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right).$$

Find the change-of-coordinates matrix $P_{\mathcal{B}\leftarrow\mathcal{C}}$. [5 pts]

$$[1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [x]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[x^2]_{\mathcal{B}} = \begin{bmatrix} 1/3 \\ 0 \\ 2/3 \\ 0 \end{bmatrix}, \quad [x^3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3/5 \\ 0 \\ 2/5 \end{bmatrix}$$

So, $$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 2/5 \end{bmatrix}$$

(b) If $p(x) = 1 + x + x^2 + x^3$, then find $[p(x)]_{\mathcal{B}}$. [2 pts]

We have
$$[p(x)]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\,[p(x)]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 2/5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4/3 \\ 8/5 \\ 2/3 \\ 2/5 \end{bmatrix}$$