MATH 2090: Linear Algebra 2 Dr. S. Cooper, Fall 2018

Mastery Quiz 2 (B02 & B03)

Name and Student Number: Sample Solutions

In the space provided, please write your solutions to the following exercises. *Fully explain your work.* Remember to use good notation and full sentences. No resources (such as notes, texts, cell phones, calculators, translators, etc.) are permitted.

Good Luck!

1. Are the following statements true (T) or false (F)? Circle your answers. No justification is required and no partial credit will be given for each statement. [5 pts]

   (a) $\{1-3x+5x^2, -3+5x-7x^2, -4+5x-6x^2, 1-x^2\}$ is a basis for $\mathcal{P}_2(\mathbb{R})$. T F

   (b) The dimension of $M_{4\times 3}(\mathbb{R})$ is 12. T F

   (c) The set $\{0, 1+x, x^3\}$ is linearly independent in $\mathcal{P}_3(\mathbb{R})$. T F

   (d) Any five vectors in a 4-dimensional vector space are linearly independent. T F

   (e) All bases of a vector space V have the same number of vectors. T F

2. Let $\mathcal{B} = \{1 - x^2, x - x^2, 2 - x + x^2\}$ be a subset of $\mathcal{P}_2(\mathbb{R})$.

(a) Prove that $\mathcal{B}$ is linearly independent. [5 pts]

Let $a, b, c \in \mathbb{R}$ such that
$a(1-x^2) + b(x-x^2) + c(2-x+x^2) = 0 + 0x + 0x^2$.
Then by comparing coefficients we have

$$\left.\begin{array}{l} a+2c=0 \\ b-c=0 \\ a-b+c=0 \end{array}\right] \Rightarrow \begin{array}{l} a=-2c \\ b=c \end{array} \Rightarrow \begin{array}{l} 0 = a-b+c \\ \quad = -2c-c+c \\ \quad = -2c \end{array}$$

$\therefore\ c = 0 \Rightarrow a = 0, b = 0$

Thus, by definition, $\mathcal{B}$ is linearly independent.

(b) Prove that $\mathcal{B}$ is a basis for $\mathcal{P}_2(\mathbb{R})$. [2 pts]

We know that $\mathcal{B}$ is linearly independent and has 3 vectors.
Thus, since $\dim(P_2(\mathbb{R})) = 3$, $\mathcal{B}$ must be a basis for $P_2(\mathbb{R})$.

(c) Find the polynomial $q(x) \in \mathcal{P}_2(\mathbb{R})$ such that [3 pts]

$$[q(x)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$

$$\begin{aligned} q(x) &= 0(1-x^2) + 2(x-x^2) - 1(2-x+x^2) \\ &= 2x - 2x^2 - 2 + x - x^2 \\ &= -2 + 3x - 3x^2 \end{aligned}$$

3. The set

$$\mathcal{U} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = c \right\}$$

is a subspace of $M_{2\times 2}(\mathbb{R})$.

(a) Find a basis for $\mathcal{U}$ and prove that your answer is indeed a basis. [4 pts]

$$\mathcal{U} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a+d=c \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ a+d & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \right\}$$

$$= \left\{ a\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} : a,b,d \in \mathbb{R} \right\}$$

$$= \operatorname{Span}\left(\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\} \right)$$

If $x, y, z \in \mathbb{R}$ such that

$$x\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} x & y \\ x+z & z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then we must have $x = y = z = 0$

Thus, $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\} = \mathcal{B}$ is linearly independent and spans $\mathcal{U}$. We conclude that $\mathcal{B}$ is a basis for $\mathcal{U}$.

(b) What is the dimension of $\mathcal{U}$? [1 pt]

$\dim(\mathcal{U}) = 3$ by part (a).