MATH 2090: Linear Algebra 2 Dr. S. Cooper, Fall 2018

Mastery Quiz 1 (B01)

Name and Student Number: Sample Solutions

In the space provided, please write your solutions to the following exercises. *Fully explain your work.* Remember to use good notation and full sentences. No resources (such as notes, texts, cell phones, calculators, translators, etc.) are permitted.

Good Luck!

1. Are the following statements true (T) or false (F)? Circle your answers. No justification is required and no partial credit will be given for each statement. [5 pts]

   (a) Additive inverses in a vector space are sometimes not unique. T F

   (b) *Span* $\left(\left\{\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -4 & 3 \\ 10 & 5 \end{bmatrix}\right\}\right)$ is a subspace of $M_{2\times 2}(\mathbb{R})$. T F

   (c) If $\mathcal{U}$ is a subspace of the vector space V, then the zero vector of $\mathcal{U}$ is the same vector as the zero vector of V. T F

   (d) If $\mathbf{v}$ is a non-zero vector in a vector space V, then $0\mathbf{v} \neq \mathbf{0}$. T F

   (e) In a vector space, the vector $(-1)\mathbf{v}$ is the additive inverse of $\mathbf{v}$. T F

2. Let V be the set of all ordered pairs of real numbers (a, b). Define an addition for the elements of V by the rule

$$(a, b) \oplus (c, d) = (a + c, b + d),$$

and a multiplication of elements of V by real numbers with the rule

$$r \cdot (c, d) = (r + c, rd).$$

Is V with these two operations a vector space? Justify your answer. [4 pts]

No, V is not a vector space over $\mathbb{R}$.
If it were then $1 \cdot (c,d) = (c,d)$ for all $(c,d) \in V$. However,

$$1 \cdot (5,4) = (1+5, 1 \cdot 4) = (6,4) \neq (5,4)$$

3. Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define operations $\oplus$ and $\circ$ on V by $x \oplus y = xy$ and $t \circ x = x^t$ for $x, y \in V$ and scalars $t \in \mathbb{R}$. You may assume without proof that V is a vector space over $\mathbb{R}$ with these operations. Determine the zero vector of V. For full credit you must demonstrate that your claimed zero vector satisfies the necessary properties. [4 pts]

Call the zero vector y. Then for $x \in V$ we need

$$x \oplus y = y \oplus x = x.$$

That is, we need $xy = x$ and so $y = 1$.

Check: If $x \in V$ and $y = 1$, then

$$x \oplus y = xy = x(1) = x$$

and

$$y \oplus x = yx = (1)x = x.$$

4. Prove that $\mathcal{U} = \{a + bx + cx^2 : a + c = b\}$ is a subspace of $\mathcal{P}_2(\mathbb{R})$. [5 pts]

(i) $0 + 0x + 0x^2 \in \mathcal{U}$ since $0+0=0$

So $\mathcal{U}$ is non-empty.

(ii) Let $p(x) = a+bx+cx^2 \in \mathcal{U}$, $q(x) = a'+b'x+c'x^2 \in \mathcal{U}$

Then

$(p+q)(x) = (a+a') + (b+b')x + (c+c')x^2 \in P_2(\mathbb{R})$

and

$(a+a') + (c+c') = (a+c) + (a'+c') = b+b'$

Thus, $p+q \in \mathcal{U}$.

(iii) Let $\alpha \in \mathbb{R}$ and $p(x) = a+bx+cx^2 \in \mathcal{U}$.

Then $(\alpha p)(x) = \alpha a + \alpha b x + \alpha c x^2 \in P_2(\mathbb{R})$

and $\alpha a + \alpha c = \alpha(a+c) = \alpha b$

Thus, $\alpha p \in \mathcal{U}$

By the Subspace Test,

$\mathcal{U}$ is a subspace of $P_2(\mathbb{R})$.

5. Complete the following definition:

Let V be a vector space over the field $\mathbb{F}$ and let $\mathbf{A_1}, \ldots, \mathbf{A_k}$ be vectors in V. A *linear combination* of $\mathbf{A_1}, \ldots, \mathbf{A_k}$ is [2 pts]

a vector of the form

$\underline{A} = a_1 \underline{A_1} + \cdots + a_k \underline{A_k}$

where $a_1, \ldots, a_k$ are in $\mathbb{F}$.