Linear Transformations: Isomorphisms

Let's look at some further examples of injective and surjective linear transformations.

Examples:

1. Consider the linear transformation $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(p(x)) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix}.$$

2. Let $T : \mathbb{R}^2 \to \mathbb{R}$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x.$$

We now look at conditions that guarantee isomorphisms.

Theorem: A linear transformation $T : V \to W$ is an isomorphism if and only if T is injective and surjective.

Examples:

1. The function $T : M_{2\times 2}(\mathbb{C}) \to \mathbb{C}^3$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{pmatrix} a+b \\ b-2c \\ a+b+d \end{pmatrix}$$

   is a linear transformation.

2. The function $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ defined by

$$T(a+bx+cx^2) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

   is a linear transformation.

Question: The transformation T from the first example above is not an isomorphism. Can we find *any* linear transformation that is an isomorphism between $M_{2\times 2}(\mathbb{C})$ and $\mathbb{C}^3$? The following result says "no"!

Theorem: Let V and W be finite-dimensional vector spaces over the field $\mathbb{F}$. Then V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

Remark: This proof also tells us how to make the isomorphism $T : V \to W$ once we know that $\dim(V) = \dim(W)$.

Corollary: If V is any vector space over the field $\mathbb{F}$ with $\dim(V) = n$, then $V \cong \mathbb{F}^n$.

Example: $\mathcal{P}_2(\mathbb{R}) \cong \mathbb{R}^3$, $M_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^4$, but $M_{2\times 2}(\mathbb{C}) \ncong \mathbb{C}^3$.