Inner Product Spaces: Projections

Last Class: We defined the projection of a vector $\mathbf{v}$ onto a vector $\mathbf{w}$ in an inner product space. Intuitively,

Example: Let $\mathbf{v} = \begin{bmatrix} 4 \\ 1+i \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ be vectors in $\mathbb{C}^3$. Using the standard Hermitian inner product, we have

Question: How do we project onto higher-dimensional subspaces?

Definition: Let V be an inner product space and $W \subseteq V$ be a subspace. Let $\{\mathbf{w_1}, \ldots, \mathbf{w_k}\}$ be an *orthogonal* basis for W. Let $\mathbf{v} \in V$.

1. The **projection of v onto** W is

2. The **perpendicular vector of v with respect to** W is

Facts: Let V be an inner product space and W be a subspace of V. Let $\mathbf{v} \in V$.

1. For all $\mathbf{w} \in W$,
$$||\mathbf{v} - \text{proj}_W(\mathbf{v})|| \leq ||\mathbf{v} - \mathbf{w}||.$$

2.
$$||\mathbf{v} - \text{proj}_W(\mathbf{v})|| = ||\mathbf{v} - \mathbf{w}|| \implies \mathbf{w} = \text{proj}_W(\mathbf{v}).$$

So ..... $\text{proj}_W(\mathbf{v})$

Example: Consider $V = \mathcal{P}_3(\mathbb{R})$ with respect to the inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$.

We are now in position to apply projections to obtain orthonormal bases!

Gram-Schmidt Orthogonalization Procedure: Let V be an inner product space with basis $\mathcal{B} = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$. Define $\mathcal{C} = \{\mathbf{w_1}, \ldots, \mathbf{w_n}\}$ as follows:

$$\begin{aligned}
\mathbf{w_1} &= \mathbf{v_1} \\
\mathbf{w_2} &= \mathbf{v_2} - \frac{\langle \mathbf{v_2}, \mathbf{w_1} \rangle}{||\mathbf{w_1}||^2}\mathbf{w_1} = \mathbf{v_2} - \mathrm{proj}_{\mathbf{w_1}}(\mathbf{v_2}) = \mathrm{perp}_{\mathbf{w_1}}(\mathbf{v_2}) \\
\mathbf{w_3} &= \mathbf{v_3} - \frac{\langle \mathbf{v_3}, \mathbf{w_1} \rangle}{||\mathbf{w_1}||^2}\mathbf{w_1} - \frac{\langle \mathbf{v_3}, \mathbf{w_2} \rangle}{||\mathbf{w_2}||^2}\mathbf{w_2} = \mathrm{perp}_{Span(\{\mathbf{w_1}, \mathbf{w_2}\})}(\mathbf{v_3}) \\
&\vdots \\
\mathbf{w_n} &= \mathbf{v_n} - \frac{\langle \mathbf{v_n}, \mathbf{w_1} \rangle}{||\mathbf{w_1}||^2}\mathbf{w_1} - \cdots - \frac{\langle \mathbf{v_n}, \mathbf{w_{n-1}} \rangle}{||\mathbf{w_{n-1}}||^2}\mathbf{w_{n-1}} = \mathrm{perp}_{Span(\{\mathbf{w_1}, \ldots, \mathbf{w_{n-1}}\})}(\mathbf{v_n})
\end{aligned}$$

Furthermore, define $\mathcal{D} = \{\mathbf{u_1}, \ldots, \mathbf{u_n}\}$ where

$$\mathbf{u_j} = \frac{1}{||\mathbf{w_j}||}\mathbf{w_j}.$$

Proposition: With the previous notation:

1. For $1 \leq j \leq n$, we have

$$Span(\{\mathbf{v_1}, \ldots, \mathbf{v_j}\}) = Span(\{\mathbf{w_1}, \ldots, \mathbf{w_j}\}) = Span(\{\mathbf{u_1}, \ldots, \mathbf{u_j}\}).$$

2. $\mathcal{C} = \{\mathbf{w_1}, \ldots, \mathbf{w_n}\}$ is an

3. $\mathcal{D} = \{\mathbf{u_1}, \ldots, \mathbf{u_n}\}$ is an

Corollary: Every inner product space has an orthonormal basis.

Example: Consider $\mathbb{R}^3$ with respect to the usual dot product. The set

$$\mathcal{B} = \left\{ \mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^3$.

Our Next Goal: To apply projections to approximations of data functions. We need preliminaries of orthogonal complements first!

Orthogonal Complements

Observe: Every subspace W of an inner product space V has an associated subspace.

Definition: Let W be a subspace of an inner product space V. The **orthogonal complement** of W is

Example: Consider $V = \mathbb{R}^3$ with respect to the usual dot product. Let

$$W = Span\left(\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}\right).$$