Inner Product Spaces: Orthogonality

Recall: If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$ and θ is the angle between these two vectors, then

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||||\mathbf{w}||}.$$

Thus, $\mathbf{v}\cdot\mathbf{w} = 0$ implies that $\mathbf{v}$ and $\mathbf{w}$ are perpendicular.

Definition: Let V be an inner product space. We say that $\mathbf{v}$ and $\mathbf{w}$ are **orthogonal**, denoted

Example: Consider $M_{2\times 2}(\mathbb{R})$ with the inner product defined by

$$\langle A, B\rangle = \operatorname{tr}(A^T B).$$

Let

$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(Pythagorean) Theorem: Let V be an inner product space. If $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, then

$$||\mathbf{v}+\mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2.$$

Lemma: Let V be an inner product space and $\mathbf{v}, \mathbf{w} \in V$ such that $\mathbf{w} \neq \mathbf{0}$. Then $\mathbf{w}$ is orthogonal to $\mathbf{v} - \dfrac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\mathbf{w}$.

Cauchy-Schwarz Inequality: Let V be an inner product space. Then

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq ||\mathbf{v}||\ ||\mathbf{w}||.$$

The Cauchy-Schwarz Inequality is what allows us to make the following formal definition for the angle between vectors of inner product spaces over the real numbers.

Definition: Let V be a *real* inner product space. We define the **angle θ between $\mathbf{v}$ and $\mathbf{w}$ in** V by

We now list some properties of the norm of a vector in an inner product space.

Proposition: let V be an inner product space. The following properties hold for all $\mathbf{v}, \mathbf{w} \in V$ and scalars α:

1. $||\alpha \mathbf{v}|| = |\alpha| \ ||\mathbf{v}||$;
2. (**Triangle Inequality**) $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$;
3. $||\mathbf{v}|| = 0 \implies \mathbf{v} = \mathbf{0}$.

Orthonormal Bases

Goal: To find bases that behave like the standard basis for $\mathbb{R}^n$ with respect to the dot product.

Definition: Let V be an inner product space.

1. $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\} \subseteq V$ is called **orthogonal** if $\langle \mathbf{v_i}, \mathbf{v_j} \rangle = 0$ whenever $i \neq j$.
2. $\mathbf{v}$ in V is called a **unit vector** if $||\mathbf{v}|| = 1$.
3. $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\} \subseteq V$ is called **orthonormal** if

Examples:

1. Consider $M_{2\times 2}(\mathbb{R})$ with the inner product defined by

$$\langle A, B \rangle = \text{tr}(A^T B).$$

Let

$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

2. $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\} \subseteq \mathbb{R}^2$

Proposition: If $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ is an orthogonal set and $\mathbf{v_i} \neq \mathbf{0}$ for all i, then $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ is linearly independent.

Definition: $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ is called an **orthonormal basis** for an inner product space V if

Examples:

1. The standard basis for $\mathbb{C}^n$ with respect to the standard Hermitian inner product is an orthonormal basis.

2. $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

3.
$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right), \sqrt{\frac{7}{2}}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) \right\}$$

Question: Can we always find orthonormal bases? How? For this we will need something called projections!

Definition: Let V be an inner product space and $\mathbf{w}, \mathbf{v} \in V$ with $\mathbf{w} \neq \mathbf{0}$.

1. The **projection of v onto w** is

2. The **perpendicular vector of v with respect to w** is